A Lifetime Distribution Model of depreciable and reproducible Capital Assets

Bekker, P.C.F.

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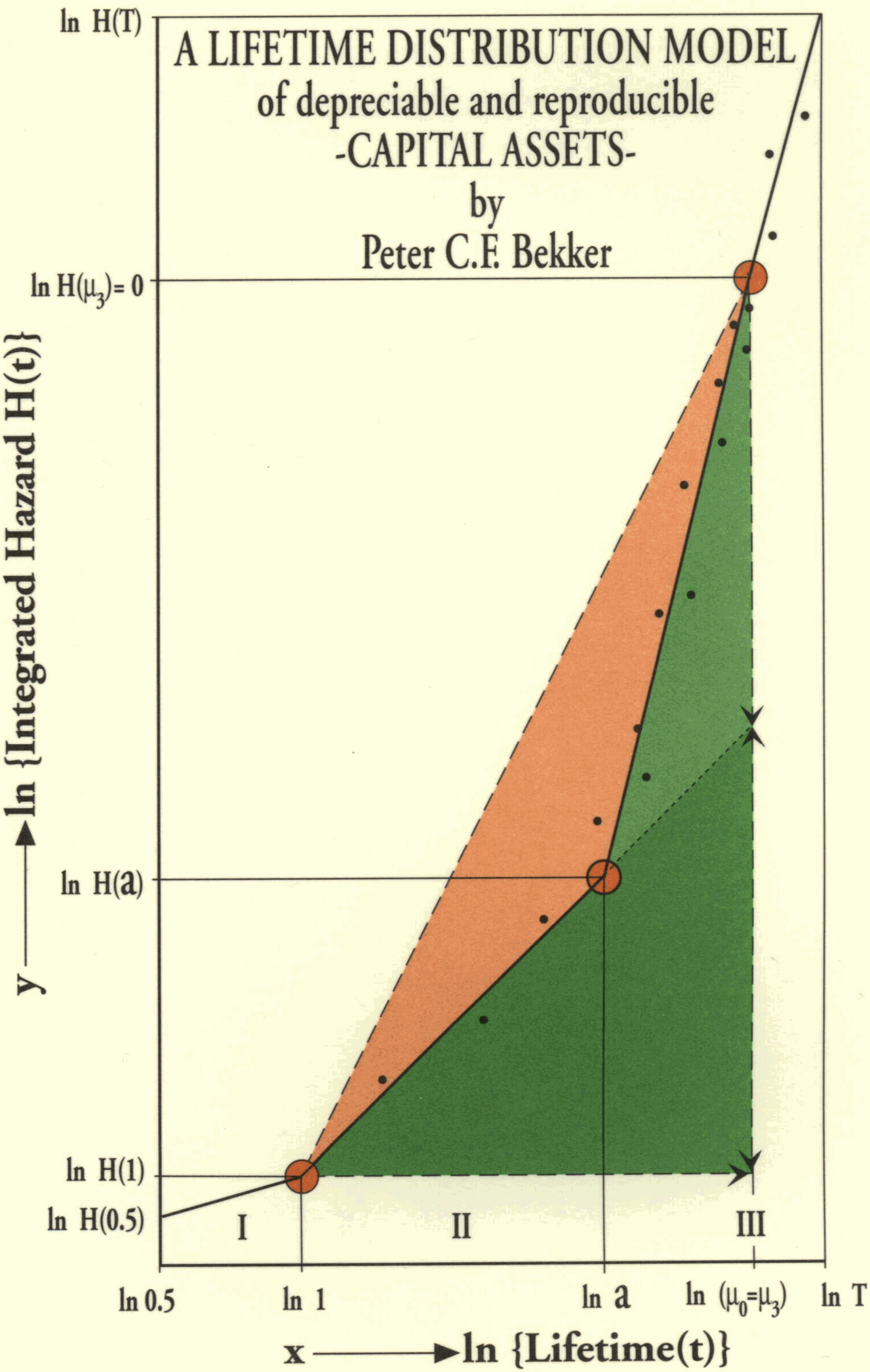
of depreciable and reproducible
-CAPITAL ASSETS-
by

Peter C.F. Bekker



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Copromotor : prof.dr. G. Ridder
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PREFACE

The motivation for this study is derived from the author's dissertation (1980) dealing with "Lifetime of Dwellings and Scarce Resources" which was prepared for the Department of Building Economics at the Slovak Technical University in Bratislava (Czechoslovakia).

In this thesis there is a further development of lifetime distribution models for depreciable and reproducible capital assets such as plant and equipment and other capital stock components. It is prepared for the Faculty of Economics at the Free University in Amsterdam (Netherlands) under the scientific supervision of Prof. Dr. A.H.Q.M. Merkies (Division of Econometrics). He drew my attention to the lack of knowledge about the subject concerned and has stimulated and guided this study. A significant contribution to the supervision and guidance in this study was given by Prof. Dr. G. Ridder of the Department of Economics, Division of Econometrics, at Groningen University. His work and knowledge of statistical analysis of the duration of unemployment in labour market experience, RIDDER (1987), were very helpful.

"Survivor functions" are needed for measuring capital stock and for all kind of econometric work in which lifetime distributions are involved. The classic work conducted by ROBLEY WINFREY (1931/1935) during the 1920's and 1930's is still widely used. He constructed empirically eighteen different (generalized) survival curves for what in his publication are called "property goods". To the best of our knowledge, none or only fragmentary empirical work on the development of survivor functions of capital assets has been published since 1935.

However, in the meantime considerable work has been carried out on lifetime theory in the medical, biochemical and technical field. This is reflected by an extensive literature which has proved to be valuable and has deepened my insight into further development of lifetime distribution models for depreciable and reproducible capital assets and manufactured durables.

As encountered in many econometric models in which lifetime functions are involved, an arbitrary (parametric) lifetime distribution is assumed or simulated because survival data and statistical information are very sparse. Taking into consideration the many categories of capital assets, their totally different utilization, their states of aggregation and their long service life (25 years and more for plant and equipment, up to centuries for certain buildings and other civil assets), a lack of proper data is not surprising. The latter limits tests of validity for the theoretical lifetime models. For the same reason, it is difficult to construct survivor curves solely on empirical grounds as was done by WINFREY. It needs to be supported by knowledge about the discarding process.

This dissertation may be regarded as illustrating a theoretical approach which aims at providing a complementary tool for modelling lifetime distributions and associated depreciation (capital consumption) patterns of depreciable and reproducible capital assets and manufactured durables.

ABSTRACT

According to the usual discarding rule, the optimal economic lifespan of capital assets and manufactured durables is attained at the point in service when the marginal revenues are equal to the marginal costs. To cover all discarding determinants including those which cannot be expressed in economic terms, an unequivocal performance-rate indicator was employed. The discarding rule applied in this study starts from a continuous process of deliberation about whether or not to discard in the face of uncertainty with respect to a performance rate which changes with time.

Performance and lifetime affect capital consumption comprising the (net) investment and an economic provision to maintain the production and/or service function in the most economic manner. The maintenance provision per unit of time is reflected by a function that increases with time due to economic aging and technical wear and tear. Since the average depletion of the (net) investment per unit of time decreases with time, the total average capital consumption per unit of time is represented by a U-shaped curve. The average total capital consumption is minimized at the point in time which is termed "characteristic lifetime".

A probabilistic approach towards the average total capital consumption per unit of time led to a robust hazard function. It was shown that the hazard rate associated with the minimum average total capital consumption is equal to the inverse of the characteristic lifetime. This result was also obtained on the basis of a WEIBULL distribution taken as a working hypothesis. The integrated hazard at the characteristic lifetime appeared as an elasticity equal to the inverse WEIBULL shape parameter for which a number of (characteristic) values are theoretically derived.

A 3-component (composite) lifetime distribution model was constructed on the basis of three distinctive risk-specific WEIBULL hazard concepts characterized by a decreasing, a constant and an increasing hazard rate for subpopulations which fail independently during Phase I, II or III of their service life respectively. The specification of the model concerned is such that the integrated hazard pattern is convexly increasing during Phase I, linearly increasing during Phase II and progressively increasing with time during Phase III. The model was tested by means of empirical retirement data on 96 different sets representing a large variety of capital assets and manufactured durables.

Next the model was applied to demonstrate that the depreciation/capital consumption ratio is equal to the integrated hazard, thus identical to the minus-log probability of survival at every point in time. That ratio is one when the lifetime is equal to the WEIBULL size parameter which is a crucial value in the light of technological progress and capital consumption. Our depreciation methodology was compared with two other methods.

Some relevant deterministic replacement models were discussed and interpreted with respect to the model theoretically developed. Finally, attention was paid to apply the inverse WEIBULL shape parameter as a capital elasticity, e.g., in the COBB-DOUGLAS production function and in the empirical Learning Curve function.

SAMENVATTING (Abstract in Dutch)

Overeenkomstig de gebruikelijke afstotingsregel wordt de optimale economische gebruiksduur van kapitaalgoederen en van duurzame industriële produkten bereikt op het moment dat de marginale opbrengst gelijk is aan de marginale kosten. Om alle beslissende afstotingsfaktoren te laten meespelen, ook die welke niet in economische termen kunnen worden uitgedrukt, werd een éénduidige prestatie-indicator gebruikt. De in deze studie toegepaste afstotingsregel gaat uit van een voortdurend afwegingsproces tussen afstoten of niet, in samenhang met de onzekerheid wat betreft het verloop van een in de tijd veranderende prestatie-indicator.

Prestatie en gebruiksduur zijn van invloed op het kapitaalgebruik dat bestaat uit de (netto) investering plus een economische voorziening om de produktie- en/of dienstverlenende funktie in stand te houden op de meest rendabele wijze. Deze instandhoudingsvoorziening per tijdseenheid wordt weergegeven door een funktie welke toeneemt met de gebruiksduur vanwege economische veroudering en technische slijtage. Omdat het gemiddelde verbruik van de (netto) investering per tijdseenheid afneemt met de gebruiksduur, wordt het gemiddelde totale kapitaalgebruik weergegeven door een U-vormige curve. Het gemiddelde, totale kapitaalgebruik is minimaal op het tijdstip dat de karakteristieke gebruiksduur genoemd wordt. Een waarschijnlijkheidsbenadering met betrekking tot het gemiddelde, totale kapitaalgebruik per tijdseenheid leidde tot een ongecompliceerde hazard-funktie. Aangetoond werd dat de "hazard rate", behorende bij het laagste gemiddelde (totale) kapitaalgebruik, gelijk is aan de omgekeerde karakteristieke gebruiksduur. Hetzelfde resultaat werd ook verkregen op basis van een WEIBULL verdeling waarvan als werkhypothese werd uitgegaan. De "integrated hazard" bij de karakteristieke gebruiksduur bleek een elasticiteit te zijn, welke gelijk is aan de omgekeerde WEIBULL vormparameter, waarvoor een aantal (karakteristieke) waarden theoretisch werden afgeleid.

Een 3-delig (composiet) model voor de verdeling van gebruiksduren werd opgesteld op basis van 3 verschillende, afstoting-specifieke WEIBULL hazard-concepten welke kenmerkend zijn voor een afnemende, een constante of een toenemende "hazard rate" met betrekking tot subpopulaties die, onafhankelijk van elkaar, afgestoten worden gedurende resp. Fase I, II of III van hun gebruik. De specificatie van het model is zodanig, dat de continue "integrated hazard" curve degressief in de tijd oploopt in Fase I, lineair stijgt in Fase II en progressief oploopt in Fase III. Het model werd getoetst aan de hand van opgespoorde, empirische afstotingsgegevens van een 96-tal representatieve kapitaalgoederen en duurzame industriële produkten van grote verscheidenheid.

Vervolgens werd het model toegepast om aan te tonen dat de verhouding afschrijving/kapitaalgebruik gelijk is aan de "integrated hazard" en dus aan de minus-log overlevingskans op elk tijdstip. Die verhouding is één wanneer de gebruiksduur gelijk is aan de WEIBULL grootteparameter, hetgeen een cruciale waarde is in het licht van technologische vooruitgang en kapitaalverbruik. Onze afschrijvingsmethodologie werd vergeleken met 2 andere methoden.
Enige relevante deterministische vervangingsmodellen werden besproken en geïnterpreteerd met betrekking tot het theoretisch opgestelde model. Tenslotte werd er aandacht besteed aan de toepasing van de omgekeerde WEIBULL vormparameter als een kapitaalelasticiteit b.v. in de COBB-DOUGLAS produktiefunktie en in de empirische funktie der Leercurve.

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CHAPTER I

GENERAL INTRODUCTION

I.1. Scope and objective

Lifetime is a key variable in several kinds of econometric models, e.g., it must be known to compute and aggregate all acquisition and other costs associated with owning and utilizing capital assets. One of the problems to be addressed by means of econometric models may be the value pattern of capital assets over time.
In theory, the value of capital can simply be defined as the discounted future income flow derived from it. Estimation of future income at a given rate of discount requires, however, complete information on performance, costs and revenues over time. This means that knowledge of lifetime characteristics is indispensable for that sort of calculations.
In spite of the extensive literature on replacement models and on methods for measuring the value of capital stock or consumption of capital, the lifetime domain in economy is still more or less undeveloped. Lack of statistical information and knowledge about the process that generates lifetime distributions are, obviously, the reasons that frequently lifetime distributions of capital assets are arbitrarily assumed. Of course, this situation is far from satisfactory and has been the incentive for the work ahead.
For the purpose of this study, a capital asset is defined as a tangible operating production or service system, being a manufactured investment good and a depreciable and reproducible component of capital stock.

Within the scope of this definition come capital assets such as:
- buildings, dwellings and all kind of civil and infrastructural assets;
- plant, machinery, equipment and installations in the manufacturing sector;
- motor vehicles (buses, commercial and passenger cars);
- aircrafts, railway and other transportation equipment;
- power stations including their networks for electricity distribution (transformers, switch stations, cables and power lines);
- machinery and equipment in the agricultural sector;

- ships (commercial and fishery fleet);
- cranes, forktrucks, excavators, and mining, dredging and road building equipment;
- public supply systems (water, gas, communication networks, etc.);
- computers and their networks;
- all kind of productive machinery and service installations in the public and private sector.

Considering the assets listed above, it is noticed that they are, generally, repairable and maintainable as they can be replaced partially or entirely. Replacement at the end of their life, however, is not a condition as such.

Property goods like land, mining resources, livestocks of the agricultural and fishery sector, producing crops and trees, etc., are excluded because they are not manufactured and non-reproducible. Stocks of non-depreciable or non-productive ware, goods or products are excluded as well. Productive depreciable and reproducible (tangible) components of the capital stock as represented in National Accounts come within the scope of this study and are covered by our definition. Various definitions as encountered in the literature will not be discussed here. When we refer in this study to "capital assets", we will always mean the one covered by our definition given above.

In this study, lifetime is defined as the length of the utilization period from the beginning of a capital asset's production or service function until it is obsolete and discarded from the class of stock in question.

Our definition of lifetime holds, no matter by which cause a capital asset is discarded from stock. Consequently, the terminology in the literature used for technical, functional, social, economic, actual or real lifetime, is irrelevant. Whatever the reason may be to withdraw an asset from capital stock, nearly all discards are directly or indirectly governed by an economic decision-making process. As a matter of fact, the lifetime of repairable assets is, technically speaking, infinite. Economically speaking, however, an asset will be discarded from use when the desired level of profitability or performance is no longer achieved. In this view, the scope of this study is economic.

Basically, lifetime is measured in units of time, however, measuring in time-related units of output, performance or otherwise is not excluded. The lifetime of machinery such as engines is frequently measured in operating hours; the life span of certain production equipment may be measured in time-related units of output, e.g. cars, lorries and rolling railway equipment, is often expressed in miles or kilometers. Also the number of "shocks" (life threatening and lifetime reducing events) is encountered in the literature for measuring any time-related life span. It may be clear that in all cases some relevant conversion to appropriate units of time will be necessary.

The measuring or estimation of the gross and/or net value of the national capital stock or any other capital stock in the public or private sector is, in itself, out of this dissertation's scope. Nevertheless, capital consumption concepts related to lifetime will be discussed and elaborated.

The objective is to extend our knowledge on lifetime characteristics of previously defined capital assets, and to facilitate modelling of their lifetime distributions to be used mainly as a complimentary tool in economics.

I.2. The Problem of Lifetime Characteristics

The lack of proper data and information on lifetime is the main reason why modelling of lifetime distributions for capital assets is necessary. To demonstrate the problem, we may consider a plant as an asset of a certain category in a certain sector of the economy. This plant is a tangible production system comprizing several subsystems which are also capital assets of various kinds. Generally, the subsystems can either be repaired/maintained or replaced at the end of each individual life. As long as this maintenance or replacement process with regard to its subsystems is going on, the entire plant is kept alive until it becomes economically obsolete. This implies that the lifetime of plants can range over 25 years up to 50 years and even longer, however, a decreasing tendency can be observed due to rapid technological change. Nevertheless, it can hardly be expected that a complete registration of failure and survival data of all plant's subsystems will be available over such a long period of time.

Moreover, plants belonging to the same sector of the economy are poles apart. Even plants of the same category in one and the same manufacturing sector will differ widely as the same plants are not necessary identical. Another statistical problem is that we are not in the position to take failure and survival records from a group of nominally identical plants. This can only occur for plants that started their production or service function at the same time as the utilization characteristics are known or, preferable, similar. This situation differs completely in comparison with reliability testing, particularly, in the technical field. For instance, a group of randomly sampled, nominally identical private cars out of a production stream of a car manufacturing plant can be tested technically until they are unfit for further use. Statistical data will lead simply to survival functions. Apart from the fact that the car manufacturer is keen on the time or performance interval between the initial start and the first failure in view of the guarantee period, the survival function obtained by such tests does not correspond with the "real world" survival function in which we are interested in this study. Due to legal registration of road vehicles like buses, lorries, commercial and private cars, and their relative short life in comparison with plants and durable equipment, their service life is recorded. This may be reason why studies on survival functions of motor vehicles are available; they will be discussed later.

In several countries ships (commercial and fishery fleet) are legally registered like road vehicles. Although this registration can be a useful source, statistical information on lifetime data is mostly not available to suite our purpose. The same is true for dwellings which are registered as well. As in the case of ships, the problem is that not all ships or dwellings are identical. In the Netherlands the age of dwellings is registered as well as discards from stock and the reason why they are discarded (obsolescence, catastrophic causes such as fire, storm, explosions, etc., infrastructural reconstruction and changing destination from dwelling to office, shop, bar, etc.). Again, this is a valuable source of information but an incomplete one. Also here we meet the problem that no straightforward statistical data are available which are obtained from a large group of nominally identical dwellings built in the same year. The average lifetime of durable dwellings is very long (70 to 90 years) as many range over centuries which creates an additional problem in view of the reliability of old and incomplete statistical data.

The age of many civil assets such as bridges, tunnels, pumping stations, etc., and public buildings like post offices, governmental offices, schools, etc., is also known, however, data and information are far from complete. The main problem here is the small number and the fact that few or none of these assets are identical.

One would expect public or semi-public bodies to have adequate statistical data concerning lifetimes of their assets (e.g. electricity generation and distribution, gas and water supply and communication systems, railways and trains). This is, however, not always the case. If any data are available, they are mainly concerned with subsystems. Also in this case, it is almost never straightforward and well documented information. This unresolved matter constitutes a great challenge to explore by means of a modelling approach.

Although insurance companies (must) deal with survival functions of capital assets, it appeared that these companies have hardly any lifetime data of capital assets at their disposal in the form we need them. The main reason is that they are more interested in catastrophic risks.

Another possible source of lifetime data may be found in tax regulations. Fiscal authorites employ, in general, a sort of mean lifetime called depreciation time, specified in guidelines and rules for different groups and types of capital assets. These depreciation times are often based on historical investments and not useful for our purposes. Probably, the most comprehensive lifetime and maintenance data are available in the aircraft sector.

The capital stock as represented in national accounts must also be ruled out as a possible source of lifetime information. The problem here is that capital assets are highly aggregated to a few distinct categories and sectors of economy. Nevertheless, in some countries attempts are being made to gather straightforward lifetime data on capital assets. By means of a lifetime database it is aimed to construct empirical survival curves as was done by ROBLEY WINFREY (1935) in the 1920's and 1930's. Even if an appropriate lifetime data bank should become available, the problem of the underlying process that generates lifetime distributions of capital assets remains. This study is an attempt to tackle the latter problem because this may fill the gap in completing our view of lifetimes that reaches beyond curve fitting of incomplete, unreliable, limited or censored lifetime data.

In spite of the lifetime documentation problem described above, we have 4 valuable sources containing 96 sets of empirical retirement data. Although these data are more or less raw, they were adequate to test the goodness of fit of the model to be constructed.

I.3. Structure of this study

This study contains 6 chapters and a number of appendices with lists and tables on empirical retirement data sets, and plots which resulted from testing. Chapters II to V begin with an introduction in which their contents are described. The collection of these 4 successive introductions may be regarded as the entire introduction to the core of this study.

Chapter II deals with starting points and mathematics needed for the construction of a probabilistic lifetime model. Since the WEIBULL distribution plays an essential role in this study, much attention is paid to its relevant (hazard) characteristics and associated functions. In that chapter the emphasis is on capital consumption and a continuous deliberation about the maintenance or discarding of capital assets and manufactured durables. In this respect deliberation is an economic decision-making process associated with uncertainty that is tackled by a probabilistic approach. Capital consumption in a probabilistic context is the core of Chapter II.

Chapter III is devoted to the construction of a probabilistic lifetime distribution model. A population mass of capital assets which have an identical production or service function and which operate independently is regarded as a collection of subpopulations. It is assumed that each subpopulation has its own risk-specific hazard characteristics which lead to a n-component (composite) distribution. The core distribution resulting from an increasing hazard rate constitutes the basis of the model.

Chapter IV emphasizes the testing procedures and the testing of the model constructed in Chapter III. For this purpose 96 empirical retirement data sets are used. These sets refer to a variety of capital assets and manufactured durables. Since these data, in general, are poorly documented, much diagnostic work is done, including simulation of measurement errors. The testing procedures which are used, are a combination of graphical analyses, data segregation techniques, the application of two different parameter estimation methods, and a check on misspecification by means of residual plotting and other techniques.

Chapter V deals with a depreciation (capital consumption) methodology resulting from the application of the findings in the foregoing chapters. The core of this chapter is the elaboration of a cumulative depreciation ratio that increases with time. Attention is paid to the economic significance of the integrated hazard parameters. The depreciation methodology developed in this chapter is compared with two other methods, one developed by the US BUREAU OF LABOR STATISTICS (1979) and the other an amortization-based method.

Chapter VI is a supplement reviewing and interpreting some relevant economic and deterministic replacement models. Suggestions for further work are indicated with respect to parameter research and investigations on capital elasticity. A first attempt is made with reference to the COBB-DOUGLAS production function and the empirical learning/experience/progress/curve function as developed by WRIGHT (1936). Finally, Chapter VI contains the summary and findings of the study ahead.

Formulae are indicated by a numerical identification (1), (2) ..... (i) in each chapter where they appear. When we refer to a formula which originates from another chapter, the Roman chapter number is added, e.g., formula, function or equation (II/7) is identified as (7) originated from Chapter II. Tables are indicated by a double numerical identification e.g., Table IV-10 is identified as Table 10 in Chapter IV. Figures are serially numbered, 1 to 17. Sections are indicated by a code that starts with the Roman chapter number followed by a serial number. Subsections have a code with more than one serial section number. Section VI.2. is the second section of Chapter VI, and Section VI.2.1.2. is the second sub-subsection of the first subsection of Section IV.2.. When we refer to literature, the name of the author(s), report or source is written in capital letters followed by the year of publication between brackets.

CHAPTER II

STARTING POINTS

II.1. Introduction

Obviously, capital assets or manufactured durables should accomplish their production and/or service function in the most economic manner. Discarding due to economic obsolescence is an economic decision. Consequently, the timing may depend on economic determinants. Optimally, marginal revenues are equal to marginal costs which can be regarded as a discarding rule. Within an interval of time, this rule involves the ratio of the value of goods or services produced (total output) to the value of the resources consumed (total input). In this view the discarding rule becomes an output/input ratio termed "productivity". In the next both the terms, productivity and performance, will be used frequently as well as output and input.

In the field of deterministic replacement models, numerous authors attribute discarding to output and/or input decay. FELDSTEIN and ROTHSCHILD (1974) provide definitions of input and output decay which are two different forms of deterioration. First, as a capital asset ages it may yield less output which is termed "output decay". Second, an older machine or piece of productive equipment may absorb more inputs while maintaining the output on the original level, which is termed "input decay". In our view it is difficult to draw distinctions between output decay and input decay. With respect to the industrial sector it is self-evident that the newest machinery and equipment consume less inputs per unit of output in comparison with older ones. In addition, the value per unit of output yielded by the newer ones may be higher because of higher quality products and/or services. Technological progress is in many cases the main driving force of the discarding process. The latter can be observed in practically all sectors where reproducible and depreciable capital assets are in use. The state of the art in technology has its own lifetime which may be regarded as a random variable governed by the process of technological research and development. From the literature it is plausible that technological progress has a significant impact on performance. But what is performance in respect of capital assets? Section II.2. deals with a productivity/performance relation framework and definitions which result in an unequivocal (discarding) indicator.

Section II.3. is devoted to fundamental starting points in relation to capital consumption and maintenance. In the context of this study capital consumption arises from decay and discarding, and from the requirement to maintain the capital stock permanently in the condition under which its function will be continued in the most economic manner. In this respect maintenance is related to capital consumption, as shown in Section II.3.1., and to depreciation as will be demonstrated later. Section II.3.2. contains basic formulae related to what is termed "average capital consumption" in connection with the probability of survival of capital assets.

Starting points concerning the probability of survival are discussed in Section II.4.. Attention is paid to different risk-specific failure (discarding) processes resulting in different kind of parameters and/or lifetime distributions. Section II.4.1. deals with the generalized GAMMA family of distributions. An essential one for this study is the WEIBULL distribution. Its relevant properties and formulae are given in Section II.4.2..

Since the so-called elasticity of probability density functions is regarded as a useful property to interpret lifetime distributions as a result of the underlying failure (discarding) process, Section II.5. deals with that subject in relation to a WEIBULL distribution.

The point in time when the average capital consumption associated with possessing and utilizing capital assets attains its minimum value, plays an important role in this study. That subject is emphasized in Section II.6.
Chapter II is completed by Section II.7 in which relevant hazard relationships are derived.

II.2. Productivity and Performance

JOHNSON A. EDOSOMWAN (1987) gives a comprehensive overview of basic definitions of productivity and types of productivity measurement dating from 1776 onwards. One major similarity that could be inferred from these various definitions is that most authors viewed productivity as a ratio of output to one input, two inputs, or total input. The measure also pertains to how well resources are utilized. Three forms of productivity are presented as follows:

1. Total productivity is "the ratio of total output to all input factors".
2. Total factor productivity is "the ratio of total output to the sum of associated labour and capital (factor) inputs".
3. Partial productivity is "the ratio of total output to one class of input".

The reader who is interested in productivity and productivity measurements is referred to SINK (1985), SUMANTH (1987) and SON & PARK (1987). The definition is as adapted to our purposes as follows:
"Productivity is the economic value of goods or services produced (total output) by means of a reproducible and depreciable capital asset, divided by the economic value of the resources consumed (total input) during the time span taken into consideration".
The economic values mentioned in this definition must be measurable which is not always easy as can be derived from the relevant literature. Furthermore, there is a need to know how efficiently resources are used, and how effective the output is as compared with input and output standards respectively. These standards are technologically, economically, ecologically, and socially in a state of flux.

STEWART (1978) defines productivity as a ratio of performance towards organizational objectives to the totality of input parameters. His definition includes performance, but this does not provide the concept required for our study. Although performance is often used in different ways and in vague terms, it seems to be a meaningful expression to indicate the ability with respect to satisfy economic, technical, social, environmental and/or other requirements. Productivity may be interpreted as a measure of performance, but this is not satisfactory for a scientific approach. Therefore, performance has to be defined precisely whereas the relationship between performance and productivity must be clarified. This relationship is shown below in Figure 1. It is a diagrammatic representation of the relationships which we shall employ to define the real (total or integrated) productivity and the performance rate of capital assets.

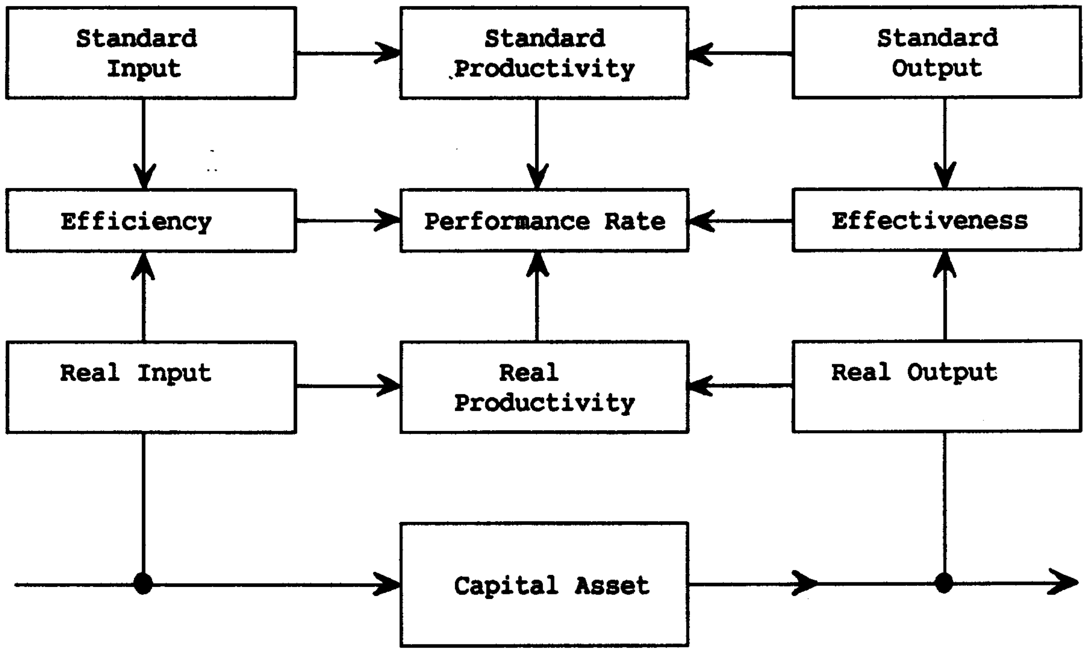


Fig. 1: Diagrammatic representation of the Productivity/Performance relation framework.

Only two quantities are measured or estimated: real input and real output during a specified time span. Then the real productivity can be determined as the ratio of real output to real input. Next the real input and the real output are compared with the standard input and standard output respectively. That calculation indicates the efficiency of the conversion process on the input side and the effectiveness on the output side. The ratio of standard output to standard input is termed "standard productivity". The ratio of real productivity to standard productivity is defined as the "performance rate" which is identical to efficiency times effectiveness. In this framework productivity and performance rate are interrelated and compared with (time-dynamic) standards. The relationships for efficiency and effectiveness are derived from VAN REEKEN (1987) who developed a measuring system for the evaluation of quality. From the explanation above the following brief definitions can be derived:

- **REAL PRODUCTIVITY** is defined as real output over real input (> 1)
- **EFFICIENCY** is defined as standard input over real input (< 1)
- **EFFECTIVENESS** is defined as real output over standard output (< 1)

Then it follows that:

- **STANDARD PRODUCTIVITY** is defined as standard output over standard input (> 1)
- **PERFORMANCE RATE** is defined as:
  . real productivity (measured during a given period of time) over standard productivity (< 1), equivalent to:
  . EFFICIENCY times EFFECTIVENESS

It is stressed that this measuring and evaluation methodology is a universal one for capital assets. It is no longer necessary to measure input and/or output in monetary terms. The only requirement is that real input and standard input are measured in equivalent units for the calculation of a dimensionless efficiency. Also on the output side, real and standard output must be measured in equivalent units such that a dimensionless effectiveness figure may be obtained.

With respect to manufactured durables such as TV-sets, washing machines, private cars, etc. in service, the difficulty is not what the costs of utilization are but what the revenue is, if any. Revenue then has to be expressed in terms of satisfaction to the user. Sometimes, the input cannot be expressed in economic terms but in terms of sacrifice whatever this may be, e.g., loss of space, loss of safety, loss of ecofunctions, pollution, etc. Then productivity is also expressed as the ratio of satisfaction (service output) to sacrifice (inputs).
However, an economic evaluation is probably impossible or at least troublesome. The introduction of the performance rate may solve that problem to a great extent. Meanwhile the methodology concerned is successfully implemented in common practice. The performance rate appears to be an adequate indicator for evaluation and control of industrial operations and of maintenance work concerning machinery and equipment, civil and infrastructural objects, motor vehicles, trains, ships, aircrafts, etc.

On closer examination of the performance rate (i.e., the product of efficiency times effectiveness, equal to the ratio of real productivity to standard productivity), it seems to be reasonable that this rate is directly related to the probability of survival. As long as that rate is sufficiently high, discarding is unlikely (barring catastrophic events). The probability of discarding increases when the performance rate decreases. Finally, the performance rate will attain the level at which the item is obsolescent. Obsolete capital assets are withdrawn from the corresponding stock and converted to a lower-valued stock. They could be put to some other, possibly less demanding use where they start a new productive life. Otherwise they are scrapped which means a conversion to the (growing) stock of waste.

The universal measurement and evaluation methodology described above provides sound definitions for the study ahead. The productivity/performance-rate concept has been chosen as a point of departure in justifying discard due to economic obsolescence. For that purpose the performance rate may be regarded as an unequivocal (discarding) indicator.

II.3. Average Capital Consumption

The possession and utilization of capital assets are associated with capital consumption. Capital consumption is the consequence of the initial investment and, in addition, of the need to maintain a capital asset in a condition to satisfy a specified performance rate. The initial investment consists of the purchase price plus the expenditure involved in the set-up and commissioning, the amount needed for dismantling or scrapping, minus the resale or scrap value at the end of a capital asset's life.

From a source depletion point of view and in regard to lifetime characteristics, we are interested in the capital consumption pattern and in the total amount of money to accomplish a given function by means of that capital asset. Generally speaking, the initial investment is known or can be estimated but the amount to maintain a capital asset in the required economic condition is unknown or, at least, uncertain as is the lifetime of the asset.

As a first step in solving the problem on hand, "average capital consumption" (per unit of time) can be defined as:

$$Y_c(T) = C(T)/T \tag{1}$$

where:

$Y_c(T)$ = average capital consumption over the effective lifespan T

$C(T)$ = amount of resources consumed in time interval $(0,T)$

T = a given (effective) lifespan

The total amount, $C(T)$, consists of two quantities. That is to say, a fixed quantity involved in the initial investment, and a quantity increasing with time needed to maintain the performance rate on a specified level at every point in time:

$$C(T) = I + M(T) \tag{2}$$

where:

I = (net) fixed quantity involved in the initial investment (in constant prices)

$M(T)$ = maintenance quantity (in constant prices) spent or set aside to compensate for a decreasing probability of survival in time interval $(0,T)$

The maintenance concept and its mathematical form are elaborated below.

II.3.1. Maintenance

In engineering practice maintenance is associated with repairable (technical) systems and subsystems. Generally speaking, such systems are reproducible and productive in manufacturing, service or use. They may include production machinery and equipment, power stations, road vehicles and rolling stock, ships and aircrafts and even durables like TV-sets, washing machines, freezers and anything of that sort as well as civil assets like roads, bridges, tunnels, port equipment, sewage systems, hydraulic works, buildings and dwellings.

From a maintance engineering point of view a repairable object is maintained "as-new" by means of repair or restoration to the initial physical state. Maintenance engineering is focussed on the failure process caused by wear and tear and by sudden (technical) disruptions. The failure process here has economic consequences which have a negative effect on productivity and, hence, on the performance of the system or subsystem under consideration. Nearly all theoretical concepts which are employed in reliability and maintenance engineering are based on an investigation of the failure pattern. This pattern can be determined empirically and analyzed by statistical methods. The failure pattern reflects the reliability, which can be translated in terms of probabilistic models. These models are used for failure-time prediction. The same probabilistic approach is used in testing in many fields where reliability and survival criteria are important. There is a vast amount of literature on the subject of maintenance and reliability theory which is potentially useful here. See, for instance, literature on methods for statistical analyses of reliability and lifetime data described by MANN, SCHAFER and SINGPURWALLA (1974). More recent literature includes the theoretical work on statistical analysis of failure-time data by KALBFLEISCH and PRENTICE (1980) and on statistical models and methods for lifetime data by LAWLESS (1982). We also refer to the work of ASCHER and FEINGOLD (1984) concerning repairable systems reliability. The authors mentioned above have included in their work a comprehensive overview of the literature on the subject under consideration. Where appropriate it is referred to it in the course of this study.

Maintenance engineering emphasizes mainly physical decay and operational disruptions of capital assets and manufactured durables in service.

Hence, maintenance aims to keep the performance rate on the desired level at every point in time. Maintenance provides for all decay, no matter what name it is given. The definition for our purposes is: "Maintenance is an economic provision that compensates for a decreasing probability of survival with time".

At point $t = 0$ in time, the cost of (or the provision for) maintaining the condition to satisfy the initially specified performance rate of a capital asset is zero. Maintenance requirements have a tendency to increase with time until the cost ceases to be feasible at point $t = T$ in time. Then the capital asset in question will be discarded from its corresponding stock. Replacement by a new one ("Challenger", embodying the newest technology) will occur if a continuation of the function as has been fulfilled in manufacturing, service or use by the older one ("Defender") is required.

The maintenance requirements in time interval (0,T) may be represented by the following function:

$$M(T) = \int_0^T m(t)dt \qquad \text{for } 0 \leq t < \infty \tag{3}$$

where:

$m(t)$ = maintenance provision per unit of time as a function of time t, for $m(0) = 0$ when $t = 0$, and $m(t)$ increasing with time

t = lifetime variable

After substituting (3) into (2), the "average capital consumption" (1) becomes:

$$Y_c(T) = \frac{C(T)}{T} = \frac{I + \int_0^T m(t)dt}{T} \tag{4}$$

Since I/T decreases and $M(T)/T$ increases with time, the average capital consumption, $C(T)/T$, is represented by a U-shaped curve. To find the relationship which minimizes $C(T)/T$ at $T = t^*$, we need to differentiate (4) with respect to the single independent variable T and set the first derivative equal to zero as follows:

$$\frac{d}{dT}\{Y_c(t^*)\} = t^{*-1}.m(t^*) - t^{*-2}\{I + \int_0^{t^*} m(t)dt\} = 0$$

and thus:

$$m(t^*) = \frac{I + \int_0^{t^*} m(t)dt}{t^*} = Y_c(t^*) \tag{5}$$

In the context of our approach elaborated above, it says that the marginal (incremental) capital consumption or resource depletion equals the long-term average total capital consumption or depletion when this amount is reduced to a minimum. The point in time when the long-term average capital consumption attains its minimum value is defined as "characteristic lifetime". This subject is continued in Sections II.6. and II.7.. It is stressed that t* is neither the optimal lifetime of a single capital asset nor of a mass of capital assets.

II.3.2. Basic Formulae

When a continuation of its function is desirable, the performance rate of a Defender (existing, older capital asset) is permanently weighed against a Challenger (newest one) which is not necessarily of the same type. According to CHOW (1960) there is an almost perfect substitution between new and older assets in terms of what he calls efficiency. Referring to the stock, he counts a one time-unit old asset as one unit while the weight for an n-time-unit old asset is the ratio of its average efficiency to the efficiency of a one-time-unit old asset for the same production or service function. In fact, CHOW's one time-unit old asset refers to the latest standard used in our performance-rate based measuring methodology. A competitive performance rate starts from the principle of a perfect substitution between a Defender and a Challenger. Consequently, there is a continuous process of deliberation in fulfilling the requisite manufacturing or service function with either a Defender or Challenger under the condition that the performance rate meets the level required for the purpose under consideration. The same sort of deliberation may take place about discarding or permanent upgrading to required standards termed as maintenance in an economic sense.

Assuming an equilibrium resulting from a continuous process of deliberation on the basis of the discarding weight (initial investment I, times the cumulative probability of discarding) against the maintenance weight (additional investments in upgrading the asset, times the probability of survival), we obtain the following equality on the basis of a probabilistic approach:

$$I.F(t) = M(t).S(t) \tag{6}$$

where:

$F(t)$ = cumulative probability of discarding as a function of lifetime variable t

$S(t) = 1 - F(t)$ = probability of survival

From equation (6) it can be derived that:

$$M(t) = I\left(\frac{F(t)}{S(t)}\right) = I\left(\frac{1 - S(t)}{S(t)}\right) = \frac{I}{S(t)} - I \qquad (7)$$

Substituting (7) into (2) gives for $t = T$:

$$C(T) = I/S(T) \qquad (8)$$

After substituting (8) into (1), the "average capital consumption" is written as:

$$Y_c(T) = \frac{I}{T.S(T)} \qquad (9)$$

This is the basic formula derived by BEKKER (1980) in "Lifetime of Dwellings and Scarce Resources". The results obtained above are applied in Section II.7. where relevant hazard relationships are derived.

The next step is to introduce an appropriate probabilistic model in order to determine S(T). In the case of dwellings, BEKKER (1980) found a good curve-fit to empirical retirement data on the basis the following parametric distributions:

. ERLANG

. left-side truncated NORMAL

. LOGISTIC

. WEIBULL

Finally, the WEIBULL distribution was chosen because the curve-fit was quite good. Moreover, the latter distribution has superior properties which provide for a family of decreasing, constant and increasing hazard rates. The WEIBULL distribution seems to be appropriate as can be derived from the vast amount of publications in the field of maintenance engineering, medicine, biomedical sciences and in all kind of duration studies. In consequence of the findings of these researchers among whom BEKKER (1980), the WEIBULL distribution was chosen as a working hypothesis for modelling the lifetime of capital assets. Modelling itself is emphasized in Chapter III.

II.4. Probability of Survival Starting-Points

In this study it is taken for granted that the discarding/scrapping behaviour of owners or users of capital assets is rational and dependent on the performance rate of the asset(s) concerned. Discarding of a capital asset from its corresponding class of stock can reasonably be regarded as the consequence of the first (fatal) and only failure. The pattern of failure-times can be described in terms of a distribution. On the assumption of a continuous failure process, a relevant lifetime distribution can be characterized by its probability density function (p.d.f.).

In the case of continuous distributions, the probability density function, $f(t)$, the cumulative distribution, $F(t)$, the survival function $S(t)$, and the hazard-rate function, $h(t) = f(t)/S(t)$, give mathematically equivalent specifications of the distribution of variable t. It is easy to derive expressions for $S(t)$ and $f(t)$ in terms of $h(t)$.

Since $f(t) = F'(t) = -S'(t)$, it follows that:

$$h(t) = \frac{f(t)}{S(t)} = \frac{F'(t)}{S(t)} = \frac{-S'(t)}{S(t)} = -\frac{d}{dt}\ln S(t)$$

Integration between $t = 0$ and $t = T$ gives the so-called integrated hazard:

$$H(T) = \int_0^T h(t)dt = -\left|\ln S(t)\right|_0^T = -\ln S(T) \qquad (10)$$

because $S(0) = 1$. For continuous distributions we obtain:

$$S(T) = \exp[-\int_0^T h(t)dt] = \exp[-H(T)] \qquad (11)$$

Then, the probability of discarding at $t = T$ becomes:

$$f(T) = h(T).\exp[-H(T)] \qquad (12)$$

In Chapter IV it will be demonstrated that a great deal of capital assets and manufactured durables are subject to more than one risk-specific failure process. Each single failure process is characterized by its own hazardous behaviour. The following three distinctive situations may be considered in matching an appropriate hazard-rate concept:

1. Monotonically decreasing hazard rate which is characteristic of improving condition with time. This type of failure manifests itself shortly after time t = 0 and gradually begins to decrease during the initial period of operation or service. A good example of this type is the running-in period of plant and machinery. Another example is the familiar "infant mortality" period beset by birth defects which have to be suppressed.
2. Constant hazard rate which is characteristic of sudden change failures. The cause of change failure is attributed to unusually severe events (accidents and catastrophes) and other unpredictable environmental conditions occuring throughout the entire operating or service period. Events in a human life which cause sudden death are of the same nature.
3. Monotonically increasing hazard rate which is characteristic of decay and gradual deterioration, which is typical of the decline in the potential condition, productivity or service capability. This type of failure is age- and time-dependent. Economic aging and technical wear and tear are irreversible during this period which is the most critical part of life, including human life. Resistance to the attacking processes decreases progressively with age.

The hazard rate that agrees with one or more of these situations will result in a corresponding distribution model. MANN, SCHAFER and SINGPURWALLA (1974) and others dealt with commonly-used failure distributions which have, generally, a closed form.

The EXPONENTIAL distribution of failure-times is the only one that is connected with constant (time-independent) hazard rate.

BARLOW & MARSHALL (1965) have tabulated bounds for distributions with monotone hazard rates. As long as such distributions have approximately equal means, equal variances and $F(t) = 0$ for $t < 0$, the increasing hazard rates of the different distribution types do not differ significantly. Although the LOG-NORMAL distribution may sometimes fit the empirical retirement data quite well, it is not an appropriate model because the hazard rate increases to a maximum following a S-shaped curve, and then decreases, approaching zero as t becomes large.

WEIBULL (1951) has developed a popular family of parametric failure distributions which include decreasing, constant and increasing hazard rates. The concept is an uni-modal type bound at time $t = 0$. It includes the EXPONENTIAL distribution and provides the required degree of generality and flexibility. The WEIBULL model is an asymptotic extreme-value distribution and can also be derived from the extreme-value theory or from a generalized GAMMA distribution. The latter is carried out in the next subsection where the formulae are given for reason of convenience.

II.4.1. Lifetime Distribution Models.

A very useful three-parameter model is the generalized GAMMA distribution introduced by STACY (1962) with p.d.f.:

$$f_{GG}(t) = \frac{\beta}{\mu.\Gamma(k)} \left(\frac{t}{\mu}\right)^{k.\beta-1} .\exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right] \qquad \text{for } t \geq 0 \qquad (13)$$

where k, μ and β are all (positive) parameters, and:

$\Gamma(k)$ = Gamma function of shape parameter k

t = lifetime variable

When $\beta = 1$, we obtain the GAMMA distribution with p.d.f.:

$$f_{G}(t) = \frac{1}{\mu.\Gamma(k)} \left(\frac{t}{\mu}\right)^{k-1} .\exp\left[-\left(\frac{t}{\mu}\right)\right] \qquad \text{for } t \geq 0 \qquad (14)$$

The GAMMA distribution is a natural extension of the EXPONENTIAL distribution. It can be derived by considering the time to the k-th successive arrival in a POISSON process or, equivalently, by considering the k-fold convolution of an EXPONENTIAL distribution. The GAMMA distribution is the continuous analog of the negative binomial distribution, which can also be obtained by considering the sum of k variables with a common geometric distribution with mean μ. The hazard rate of a GAMMA distribution cannot be expressed in a simple closed form and, hence, is cumbersome to graph unless the size parameter is an integer. The hazard function of a GAMMA distribution is monotonically increasing from 0 if $k > 1$, monotonically decreasing from ∞ if $k < 1$, and in either case approaches $1/\mu$ as t becomes large. When $k = 1$, the hazard-rate is $1/\mu$ and constant. The LOG-NORMAL distribution appears as a limiting case when $k = \infty$.

The WEIBULL distribution is perhaps the most widely-used lifetime distribution model because of its appropriateness and its convenience of mathematical handling. It follows from (13) when k = 1. Hence, the p.d.f. of the WEIBULL distribution is:

$$f_W(t) = \frac{\beta}{\mu} \left(\frac{t}{\mu}\right)^{\beta-1} . \exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right] \qquad \text{for } t \geq 0 \qquad (15)$$

where:

μ = size parameter

β = shape parameter

When $\beta = k = 1$, we obtain the EXPONENTIAL distribution with p.d.f.:

$$f_E(t) = \frac{1}{\mu} . \exp\left[-\left(\frac{t}{\mu}\right)\right] \qquad \text{for } t \geq 0 \qquad (16)$$

where μ is the mean and the only parameter.

An additional parameter can be introduced to the WEIBULL and also to the EXPONENTIAL distribution to adjust the (time) scale by a location parameter T_0. Then the size parameter μ changes to $(\mu - T_0)$.

McDONALD (1984) developed a much larger family of distributions starting with two four-parameter BETA distributions (kind 1 and 2). His tree of interrelated distributions contains:

GB1 = generalized BETA distribution of the first kind

GB2 = generalized BETA distribution of the second kind

B1 = BETA distribution of the first kind

B2 = BETA distribution of the second kind

GG = generalized GAMMA distribution

SM = SINGH-MADDALA distribution

LN = LOG-NORMAL distribution

Fsk = FISK distribution (special case of SM)

EXP = EXPONENTIAL distribution

Generally, the more parameters a distribution has, the better will it fit data. Yet a BETA distribution with three or even four parameters is unfit for our purposes. The reason is that the three risk-specific failure characteristics resulting in decreasing, constant and increasing hazard rates respectively, cannot be covered by anyone of these single distributions. Therefore a multi-fold composite distribution was conceived as is shown in Chapter III and thereafter.

II.4.2. WEIBULL Formulae

Since the WEIBULL concept generally provides all that is required to achieve the objective of this study, some relevant formulae are given below. For convenience, their derivations are assumed to be known.

The survivor function can easily be derived from the p.d.f. (15):

$$S_W(t) = 1 - {}_0\int^{\infty} \frac{\beta}{\mu} \left(\frac{t}{\mu}\right)^{\beta-1} .\exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right] dt = \exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right] \tag{17}$$

where $S(0) = 1$ and $S(\infty) = 0$.

The hazard rate or conditional failure intensity function can be obtained by combining (15) and (17):

$$h_W(t) = \frac{f_W(t)}{S_W(t)} = \frac{\frac{\beta}{\mu} \left(\frac{t}{\mu}\right)^{\beta-1} .\exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right]}{\exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right]} = \frac{\beta}{\mu} \left(\frac{t}{\mu}\right)^{\beta-1} \tag{18}$$

On closer examination, the WEIBULL hazard-rate function (18) reflects the three distinctive lifetime characteristics discussed in Section II.4:

1. A decreasing hazard rate corresponds to $0 < \beta < 1$.
2. A constant hazard rate corresponds to $\beta = 1$.
3. An increasing hazard rate corresponds to $1 < \beta < \infty$.
   This case can be subdivided into:
   a) A degressively increasing hazard rate: $1 < \beta < 2$.
   b) A linearly increasing hazard rate: $\beta = 2$.
   c) A progressively increasing hazard rate (the most frequent case): $\beta > 2$.

Central Moments, ν_{rW}, of the WEIBULL distribution

The r-th moment about the origin is given by:

$$\nu_{rW} = \mu^r.\Gamma\{1 + (r/\beta)\} \tag{19}$$

Average Lifetime ($\bar{t}_W$):

$$\bar{t}_W = \mu.\Gamma\{1 + (1/\beta)\} \qquad \text{for } \mu \text{ and } \beta > 0 \tag{20}$$

The Gamma function, $\Gamma\{1+(1/\beta)\}$, is one for $\beta = \infty$ and for $\beta = 1$. The latter value corresponds to an EXPONENTIAL distribution. The Gamma function attains its minimum value (0.8856) when $\beta \approx 2.16$. The value is almost constant for the range $2 < \beta < 6$, which is shown in Table II-1 below.

β = WEIBULL shape parameter	2	3	4	5	6
$\Gamma\{1+(1/\beta)\}$	0.886	0.893	0.906	0.918	0.928

Table II-1: Numerical values of the Gamma function for a given β.

According to Table II-1, a fair estimate of the mean is:

$$\bar{t} \approx 0.9\ \mu \qquad \text{for } 2 < \beta < 6$$

Hence, $S_W(\bar{t}) = S[\mu.\Gamma\{1 + (1/\beta)\}] \approx S_W(0.9\ \mu) = \exp[-0.9^{\beta}]$

Note that: $S_W(\mu) = \exp[-1] = 0.368$,where μ is the size parameter

<u>Variance (Var_W):</u>

$$Var_W = \mu^2\left[\Gamma\{1+(2/\beta)\}-(\Gamma\{1+(1/\beta)\})^2\right] \tag{21}$$

Consequently, the coefficient of variation (v_W) is:

$$v_W = (Var_W)^{\frac{1}{2}}/\bar{t}_W = \frac{[\Gamma\{1+(2/\beta)\} - (\Gamma\{1+(1/\beta)\})^2]^{\frac{1}{2}}}{\Gamma\{1+(1/\beta)\}} \tag{22}$$

The coefficient of variation is only dependent on β. For $\beta < 1$, $v_W > 1$; for $\beta = 1$ (EXPONENTIAL distribution), $v_W = 1$ and for $\beta > 1$, $v_W < 1$ because v_W then decreases according as β increases.

The coefficient of skewness $[\nu_3/\nu_2^{3/2}]$ and the coefficient of kurtosis or peakedness $[\nu_4/\nu_2^{2}]$ for a WEIBULL distribution are tabulated by JOHNSON and KOTZ (1970). These coefficients depend only on β.

<u>Mode ($\hat{t}_W$)</u>

The WEIBULL distribution is uni-modal with mode:

$$\hat{t}_W = \mu\{1 - (1/\beta)\}^{1/\beta} \tag{23}$$

For $\beta = 1$ or $\beta < 1$ there is no mode. For $\beta = \infty$, $\hat{t}_W = \mu$. From $\beta > 20$ onwards, $\hat{t}_W \approx \mu$.

Median ($\overset{o}{t}_W$):

The median applies when F(t) = S(t) or S(t) = 0.5 which, for a WEIBULL distribution, is represented by the following expression:

$$\overset{o}{t}_W = \mu\{\ln 2\}^{1/\beta} \tag{24}$$

Symmetrical Approximation ($\hat{t}_W = \overset{o}{t}_W$)

The WEIBULL distribution is approximately symmetrical when the mode and the median are identical:

$$\mu\{1 - (1/\beta)\}^{1/\beta} = \mu\{\ln 2\}^{1/\beta}$$

Then it follows that: $\beta = 1/(1 - \ln 2) \approx 3.26$ for $\hat{t}_W = \overset{o}{t}_W$.

Rayleigh distribution ($\beta = 2$)

When $\beta = 2$, the WEIBULL distribution is identical to a RAYLEIGH distribution which can be regarded as a distribution of radial error in a plane where the errors in each axis are independent and normally distributed with equal variance and zero mean. Its probability density function involves the size parameter (σ) of both NORMAL distributions to be obtained from $\mu^2 = 2.\sigma^2$. The hazard rate of a RAYLEIGH distribution increases linearly as shown by formula (18), a special case as will be demonstrated in Chapter III and thereafter.

Note

The WEIBULL distribution arises theoretically as a limit law for the smallest of a large number of independent non-negative random variables. Here it can be assumed that the limit law is directly related to the net present value (NPV) of future revenues per unit of time generated by a capital asset. The service life comes to an end when the NPV falls below a limit value. The latter process strongly suggests a WEIBULL distribution of lifetimes that we have chosen as a working hypothesis.

II.5. Elasticity of Probability Density Functions

ESTEBAN (1978/1986) has classified probability density functions in terms of an elasticity which indicates, in the interval (x, x+dx), the rate of change of the density. He meant to define the elasticity at x, $\pi(x)$, as:

$$\pi(x) = 1 + \frac{x.f'(x)}{f(x)} \tag{25}$$

where f(x) is the p.d.f. of a given distribution. Some properties of p.d.f.'s are easier to describe with this elasticity representation. In Chapter III and V it will appear that the elasticity of a p.d.f. is useful to understand and to interpret the associated distribution of lifetimes as a result of the underlying failure (discarding) process. Substituting (15) into (25) gives:

$$\pi_W(t) = 1 + \frac{t.f'(t)}{f(t)} = 1 + \frac{t \frac{d}{dt} \{\frac{\beta}{\mu} (\frac{t}{\mu})^{\beta-1} .\exp[-(\frac{t}{\mu})^{\beta}]\}}{\frac{\beta}{\mu} (\frac{t}{\mu})^{\beta-1} .\exp[-(\frac{t}{\mu})^{\beta}]} \tag{26}$$

$$= 1 + \frac{t(\frac{\beta}{\mu^{\beta}})\{(\beta-1)t^{\beta-2}.\exp[-(t/ \mu)^{\beta}] - (\frac{\beta}{\mu^{\beta}})t^{2(\beta-1)}.\exp[-(t/\mu)^{\beta}]\}}{(\frac{\beta}{\mu^{\beta}})t^{\beta-1}.\exp[-t(\mu)^{\beta}]}$$

$$\pi_W(t) = \beta - \beta(\frac{t}{\mu})^{\beta} = \beta\{1 - (t/\mu)^{\beta}\} \tag{27}$$

The elasticity function of a WEIBULL distribution includes the following integrated hazard function which can be derived from (17):

$$H_w(t) = - \ln S(t) = (t/\mu)^{\beta} \tag{28}$$

Integrated hazard function (28) will be used to form a depreciation model which is discussed in Section V.3..

II.6. Derivation of Characteristic Lifetime

The characteristic lifetime (t^*) is defined as the point in time at which the average consumption has attained its minimum value. Then, at t^*, the marginal (incremental) consumption equals the average total consumption. At that point in time, the first derivative of function (9) with respect to the single independent variable t^* is zero:

$$Y_c'(t^*) = \frac{d}{dt^*}\left(\frac{I}{t^*.S(t^*)}\right) = 0 \qquad (29)$$

The next step is to introduce an appropriate probabilistic model. According to the working hypothesis as concluded in Section II.3.2., we apply the survivor function (17) of a WEIBULL distribution:

$$S_W(t|\mu,\beta) = \exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right] \qquad \text{for } t,\ \mu \text{ and } \beta > 0 \qquad (17)$$

After substitution of $S(t^*)$ by $S_W(t^*)$ in (29), we obtain:

$$t^* = \mu\left(\frac{1}{\beta}\right)^{1/\beta} = \mu.\beta^{-1/\beta} \qquad (30)$$

where (t^*) is the characteristic lifetime as previously defined in Section II.3.1..

Subsequently, through (28) it will be possible to determine the integrated hazard at time $t = t^*$:

$$H_W(t^*) = \left(\frac{\mu.\beta^{-1/\beta}}{\mu}\right)^{\beta} = 1/\beta \qquad (31)$$

This surprisingly simple result means according to (17) that the probability of survival at $t = t^*$ depends solely on the shape parameter of the probabilistic WEIBULL-model:

$$S_W(t^*) = \exp[-1/\beta] \qquad \text{for } \beta > 1 \qquad (32)$$

This essential result was previously found by BEKKER (1980).

From (30) and (31) it follows also that: $t^*/\mu = H_W(t^*)^{H_W(t^*)}$ (33)

Since the probability of survival at mode is:

$$S_W(\hat{t}) = \exp[-\{1-(1/\beta)\}] \qquad \text{for } \beta > 1$$

it can be concluded that $S_W(t^*) = S_W(\hat{t})$ when $\beta = 2$ (RAYLEIGH distribution), which is a special case because of its linear hazard rate. For $\beta > 2$, the probability of survival at $t = t^*$ is higher than at mode ($t = \hat{t}$).

The characteristic lifetime and the size parameter of the relevant WEIBULL distribution are highly important for our model. According to (30), the ratio:

$$t^*/\mu = \beta^{-1/\beta} \tag{34}$$

is an essential one. This is illustrated by Figure 2 below.

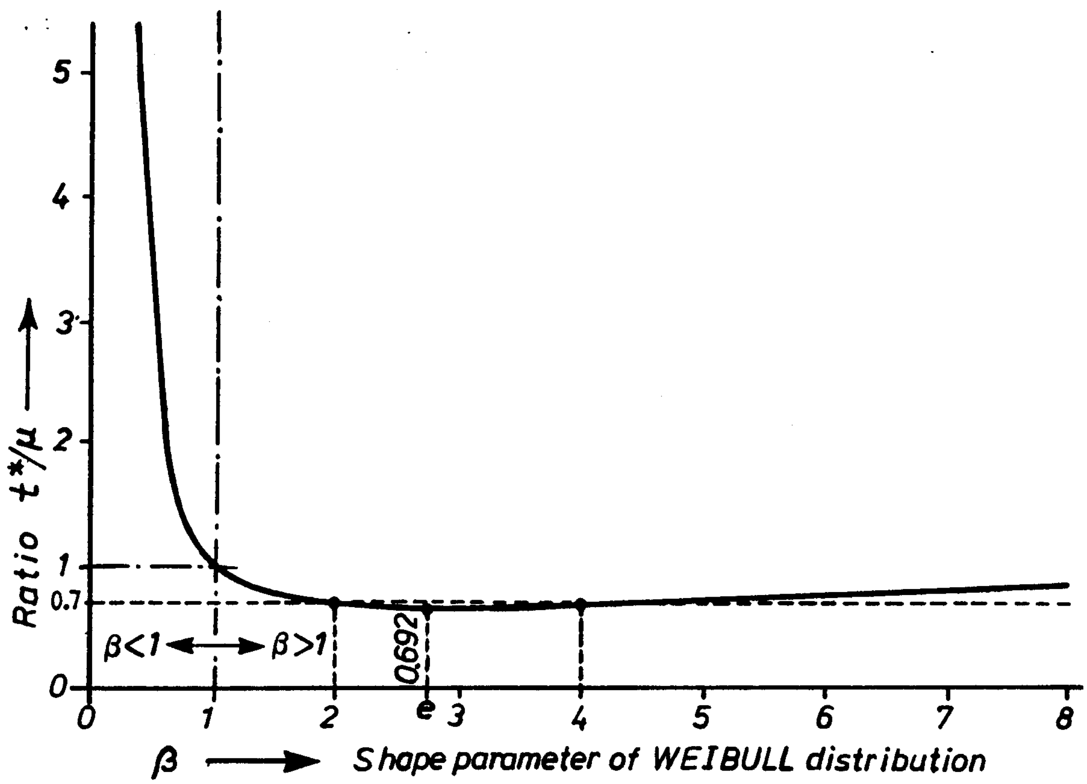


Fig. 2: Relation between the ratio (t^*/μ) and shape parameter β of a WEIBULL distribution

As shown by the curve, the ratio referred to here is almost constant for $1.6 < \beta < 6$ and identical for $\beta = 2$ and $\beta = 4$ when $t^*/\mu = 0.707$. The minimum value (0.6922) is attained when $\beta = e = 2.7183$. A fair approximation may be $t^* \approx 0.7\ \mu$.

The characteristic lifetime in the case of an EXPONENTIAL distribution proves to be $t^* = \mu$. When $0 < \beta < 1$, the characteristic lifetime is even larger than μ. Here, the EXPONENTIAL distribution can be regarded as a limiting case; then, the probability of survival at $t = t^*$ proves to be $S_E(t^*) = \exp[-1] = 0.368$.

Clearly, the shape of the distribution of lifetimes has little impact on the average capital consumption if the quantity I/μ is constant. In Chapter V it is demonstrated that the shape parameter governs the pattern of capital consumption with time.

TENGBLAD & WESTERLUND (1976) also found that the form of the lifetime distribution has only a marginal effect on capital consumption in the engineering industry. The same results were obtained by LÜTZEL (1971) who applied GAMMA distributions with different shape parameters and even a uniform distribution to estimate the capital stock in West Germany (FRG). These findings have been confirmed in more recent publications by LÜTZEL (1972/1976).

The average capital consumption according to (9) is represented by a U-shaped curve for $\beta > 1$. The slope is zero and the concavity upward at the point where the average capital consumption attains its minimum value, and when the characteristic lifetime is achieved. The level of the curve is lower and the concavity wider according as t^*, and thus μ, is higher valued, and in reverse. When I and β are given, the moment at which an asset is withdrawn from the corresponding class of stock becomes less critical according as the size parameter μ in the probability density function increases in value, and vice versa. The slope tends to become nearly horizontal over a long time interval according as t^* increases. This remarkable result can be applied in replacement decision-making and with respect to reducing the average capital consumption to a minimum.

II.7. Hazard-rate Implications

The hazard-rate function reflects the underlying process which ends in discarding. In Section II.3.1. it is made plausible that maintenance is needed to compensate for a decreasing probability of survival with time. Therefore, the hazards may arise from the maintenance process which has a tendency to increase with time.

In Section II.3.1. formulae are derived which represent the amount of capital, C(T), consumed in time interval (0,T). Combining (2) and (3) gives:

$$C(T) = I + \int_0^T m(t)dt \tag{35}$$

Differentiating (35) with respect to the independent variable T gives:

$$C'(T) = m(T) \tag{36}$$

According to (8), the amount of capital, C(T), consumed in time interval (0,T) is equal to I/S(T). Converted to its log form, we obtain:

$$-\ln S(T) = \ln C(T) - \ln I = H(T) \tag{37}$$

which is the integrated hazard at t = T. Differentiating (37) with respect to the independent variable T gives:

$$h(T) = C'(T)/C(T) \tag{38}$$

Substituting C'(T) = m(T) into (38) becomes:

$$h(T) = m(T)/C(T) \tag{39}$$

Obviously, the hazard rate is, by definition, a conditional rate that is proportional to the maintenance need at point t = T, and inversely proportional to the total amount of capital consumption in time interval (0,T).

Next, we may consider the hazard rate at the characteristic lifetime t*. Then, it follows from (5) that:

$$m(t^*) = C(t^*)/t^* \tag{40}$$

Substituting (40) into (39), gives:

$$h(t^*) = 1/t^* \tag{41}$$

Hence, the hazard rate at the characteristic lifetime is the inverse of that point in time. This is a remarkable result which is obtained independently of a particular distribution of lifetimes.

The correctness of this finding applied to a WEIBULL distribution can be demonstrated by inserting (18) and (30) in (41):

$$h_w(t^*) = \frac{\beta}{\mu}\left(\frac{\mu.\beta^{-1/\beta}}{\mu}\right)^{\beta-1} = \frac{1}{\mu.\beta^{-1/\beta}} = 1/t^* \qquad (42)$$

Both quantities are indeed equal. Herewith we have shown the hazard implications in the light of our probabilistic approach of capital consumption. The derivation of a characteristic lifetime and a hazard rate function was obtained independently of a particular distribution of lifetimes. When applied to a WEIBULL distribution, the findings of this chapter result in remarkable expressions which are summarized below:

- Hazard-rate:

$$h(T) = m(T)/C(T) \qquad (39)$$

$$h(t^*) = 1/t^* \qquad (41)$$

$$h_w(t^*) = 1/t^* \approx 1.43/\mu \quad \text{for} \quad 1.6 < \beta < 6 \qquad (43)$$

- Integrated hazard:

$$H(T) = \ln\{C(T)/I\} \qquad (37)$$

$$H_w(t^*) = 1/\beta \qquad (31)$$

- Characteristic lifetime (30) combined with (31):

$$t^* = C(t^*)/m(t^*) \qquad (44)$$

$$t^* = \mu.\beta^{-1/\beta} = \mu(1/\beta)^{1/\beta} = \mu[H_w(t^*)]^{H_w(t^*)} \qquad (45)$$

CHAPTER III

A PROBABILISTIC LIFETIME DISTRIBUTION MODEL

III.1. Introduction

To the best of our knowledge, only fragmentary work has been published on the development of universal probabilistic lifetime distribution models for capital assets since the classic work of WINFREY (1935). Some of his 18 survivor curve types are still widely used in econometric models. On closer examination of these curves, there is no closed form. The empirical hazard rates of all tabulated survivor curves tend to increase with time. It is a shortcoming of non-parametric and parametric lifetime distribution concepts employed in econometric models if these concepts are not based on distinctive life phases characterized by decreasing, constant and increasing hazard rates. It seems that the probabilistic concepts used in engineering are more sophisticated as they find their rationale in the underlying failure processes. Among the vast body of engineering science literature see, for instance, ASCHER & FEINGOLD (1984) and the references therein.

The statistical analysis of empirical retirement data of dwellings in the Netherlands, as carried out by BEKKER (1980) gives rise to the assumption that there are different hazard functions for the distinct life phases of capital assets. In the case of dwellings only two successive and distinct phases were observed, one with a constant and a second with an increasing hazard rate. A phase with a decreasing hazard rate was not established for dwellings. This might be because dwellings do not pass through the running-in period associated with complex production machinery and equipment. A dwelling might have a "debugging" period, particularly with regard to heating, ventilation and electrical installations. Also, some minor defects in the construction and finishing may have to be repaired during the first year of service, but generally these will not cause the demolition of the dwelling. In the case of durables, the manufacturer more often than not garantees the fitness for purpose according to the specifications and provides a warranty for a certain period of time, say, one year. Of course, the manufacturer will reduce the risk involved by means of repeated testing, debugging and improvement of the new product in development states until a decreasing hazard rate attains the desired value for the required purpose. Appropriate process and quality control systems ensure that manufactured goods meet the agreed specifications to fulfil their function. The relatively small risk of early failures due to initial defects is covered by the warranty. Nowadays, manufacturing is increasingly governed by product liability legislation. Finally, durables manufactured in smaller or larger batches such as passenger cars, computers, lorries, TV-sets, etc. will not generally show a decreasing hazard rate in service. Complicated capital assets such as production equipment and machinery, complex technical and operational systems and complex end products such as aircrafts and tailor-made systems of several kinds, can only be improved during the debugging and learning period when they are actually in service. This process is characterized by a decreasing hazard rate. The cost associated with the improvement of productive capital assets and manufactured durables in service will be borne, directly or indirectly, by the purchasers. In view of this, the "learning/debugging/burn-in/running-in" period takes on economic significance.

In economic terms, the life phase characterized by a constant hazard rate is most interesting because this is generally the most profitable period assuming that the constant hazard rate is sufficiently low. Obviously, this phase cannot be ignored, and this is one of the shortcomings of the WINFREY survivor curves and other single lifetime distribution models. In practice productivity (revenue over cost or satisfaction over sacrifice) can be maintained at the required level during this life phase; there is no disturbing degeneration or functional degradation in comparison with competitive goods with the same function. In other words: the performance, say vitality, meets the requirements at every point in time. The only dangers are serious events independent of time or age which cause a total loss of performance. Failure-times of this type of "one-and-only failure" (total loss of performance) are exponentially distributed. In fact, any appropriate lifetime distribution concept must (be able to) accommodate an EXPONENTIAL distribution.

The most critical part of life is the period when the potential condition or vitality, declines with time. Then the hazard rate increases monotonically which is characteristic of decay and gradual deterioration. The response to hazardous forces is eventually insufficient for survival. Nearly all probabilistic lifetime models which are employed in economics are based on this particular life phase. It is usually taken as the basis in maintenance engineering for repair/replacement cost policies and associated models. The classic **WINFREY** curves are restricted to periods of decay so that, like so many other models in use, they are limited in scope.

In Section III.2. below a few relevant lifetime distribution models are reviewed and briefly discussed. Those used in engineering, medicine and biomedical science are disregarded. The reader is referred to the vast amount of literature available in those fields.

It turns out that a universal probabilistic lifetime distribution model may properly be developed from some 3-component (composite) distribution concept akin to the three successive and distinctive life phases of depreciable and reproducible capital assets. The fundamental starting points of our concept are discussed in Section III.3..

Section III.4 deals with the mathematics of this 3-fold risk-specific lifetime distribution model followed by the elaboration of parametric relationships (Section III.4.1), the graphical representation of the modelled probability density function (Section III.4.2), a description and graphical representation of the three-component hazard concept (Section III.4.3.).

In Section II.6. we have formulated a characteristic lifetime. Subsequently, in Section III.5. characteristic shape parameters are obtained from our mathematical concept. These parameters are essential for the elaboratorion and interpretation of a suitable lifetime distribution model.

Finally, in Section III.6. the model constructed in this chapter is illustrated graphically.

III.2. Literature on Probabilistic Lifetime Distribution Models

Literature on probabilistic modelling in the field under consideration is very sparse. It is striking that not much progress has been made since the classic work by WINFREY (1931/1935) was published. In econometric practice WINFREY's empirically constructed and generalized survivor curves are the basis of what is known as Iowa-Curve Methodology (ICM). This methodology is used mainly for measuring and calculating capital stock. For this purpose it is necessary to know when assets are actually discarded from the corresponding class of stock and, in addition, how they decline in performance and value over time. But even for other purposes procedures are needed for calculating the life expectancy of industrial property which is subject to depreciation and retirement charges.

In Sections IV.7. and IV.7.1. of this thesis the original WINFREY retirement data are used to test our model. A more detailed review is therefore given in Chapter IV..

Recently DEMING & SINGPURWALLA (1989) have published critical observations on the Iowa-Curve Methodology with regard to depreciation charges. One of their criticisms deals with preselecting a particular type of generalized survivor curve for a specified application on the basis of subjective considerations. They also come to the conclusion that the family of WINFREY curves does not accomodate bathtub-shaped hazard-rate behaviour. All the curves give only an increasing hazard rate, which is limited in scope.

Nevertheless, WINFREY survivor curves are still frequently used for measuring and calculating a nation's capital stock. This is demonstrated by WARD (1976) in a report on the measurement of capital and the methodology of capital stock estimates in the OECD countries. TENGBLAD & WESTERLUND (1976) have listed 45 categories of capital assets for which so-called 3 right-modal type curves (R_1, R_2 and R_3) and 3 symmetrical type curves (S_1, S_2 and S_3) are employed by the Swedish Bureau of Statistics. As stated by DEMING & SINGPURWALLA (1989), the preselection of a particular type on the basis of subjective considerations gives rise to difficulties and is arbitrary.

The U.S. DEPARTMENT OF LABOR and the BUREAU OF LABOR STATISTICS (1979) present a methodology of capital estimation for which a so-called truncated NORMAL distribution is assumed. In contrast with the left-side truncated NORMAL distribution model as considered by BEKKER (1980) for dwellings in the Netherlands, the methodology mentioned above deals with a so-called two-sided vertically truncated and with a horizontally truncated NORMAL distribution. In both the latter cases the distribution remains symmetrical and the hazard rate increases with time. Although its scope is limited, this publication is valuable in view of the development of a depreciation function. This will be discussed in more detail in Section V.5.1..

According to LÜTZEL (1971/1972/1976) slightly left-modal GAMMA distributions are used in the Federal Republic of Germany which reflect the dispersion of scrappings and retirements over the average length of life per type of asset. The shape parameter (integer k = 9 in II/13) is given as well as the complete measuring model. LÜTZEL recognizes that his probabilistic lifetime distribution model is not determined by adequate research but derived from lifetime studies of commercial vehicles and private cars. Similar lifetime distributions are assumed for capital assets.
A similar development can be observed in Denmark where GROES (1976) adopted a lifetime distribution function for capital assets which had been found for private cars by KAERGAARD (1970). Apart from a survivor function, they also dealt with a depreciation function and claimed an exponential decay in value over time as was also found by CRAMER (1958). Therefore, according to these authors, the utility function of capital assets is a negative exponential function. This function was developed from market prices for second hand private cars (in Denmark, second hand Volkswagens). Their assumptions are very arbitrary when applied to other capital assets for which concave depreciation curves are more realistic than straight line or convex forms. This subject will be discussed exhaustively in Chapter V..

Many survival data are available with regard to motor vehicles. As a result, numerous studies have been published which deal with lifetime distributions and depreciation schemes. Many of them were examined by SMIT (1982) to estimate the future market for rubber (tyres). He analyzed and checked seven types of distributions and, in addition, developed a vintage approach model. Among the well-known distributions such as PASCAL, GAMMA, POISSON, LOGISTIC and NORMAL, only the last two were appropriate. It is interesting that the LOGISTIC lifetime distribution as developed by KIRNER (1968) had previously been used for capital stock estimation in the Federal Republic of Germany.

A great deal of capital stock consists of dwellings and built assets with extremely long service lives in comparison with cars, machinery and equipment. VERGES-ESCUIN (1981) developed a methodology for forecasting housing needs. He adopted polynomial simulations such as developed by SCHIFF (1958) for gross stock estimations from past installations. The model presents theoretical survival profiles, so-called "v"-functions. This uni-modal distribution has a closed form and is bound at both ends and offers the required degree of flexibility for purpose (exponential and from left-modal to symmetrical and right-modal).

There is a vast amount of literature on capital stock estimation but to the best of our knowledge no probabilistic lifetime distribution model has been published that provides for distinctive risk-specific survival characteristics.

There are a number of deterministic models in use for the estimation of the economic life of capital assets. Some relevant ones will be mentioned in Chapter VI..

NEWBY (1987) derived a hazard-rate and survival function for aging characteristics of an item of plant when the decision to maintain or replace is based on financial criteria. He regards the life expectancy as proportional to the net present value (NPV) of the future revenue, and demonstrated that the survival function is similar to that of the extreme value distribution. Since the extreme value distribution arises when lifetimes are taken to be WEIBULL distributed, NEWBY's finding supports the working hypothesis chosen for the construction of our lifetime distribution model.

III.3. Starting Points and Assumptions

Suppose a given population of a class (a set) of capital assets which have an identical function and which operate independently. This population is exposed to three different modes of failure processes which may result in three risk-specific hazard rates. It is assumed that the respective hazard-rate functions belong to the same family but differ in the value of their parameters.

The population can reasonably be regarded as a 3-component collection. Each component represents a subpopulation with its own hazard characteristics. The assigning of each independent unit to one of the three subpopulations can be interpreted as the probability of randomly selecting a capital unit that is predestined to fail because of a risk-specific hazardous process. This random decomposition process brings forth successively:

- Subpopulation I which fails during Phase I solely due to debugging, running-in, initial defects and early disruptions (decreasing hazard rate with time or use);
- Subpopulation II which has survived Phase I, and fails during Phase II solely due to sudden disruptions (constant hazard rate, time-independent);
- Subpopulation III which fails solely because of an increasing hazard rate with time or use due to economic aging and technical wear and tear.

The conceptual difference between a single-risk model and a 3-fold risk-specific model is that the former uses only one lifetime distribution and the latter a 3-component (composite) lifetime distribution. Apart from 3 sets of differently valued distribution parameters, a 3-component (composite) lifetime distribution has two partition parameters.

Another fundamental fact of our model is that the partition parameters are related to the risk-specific hazard-rate parameters. This is a consequence of the randomly acting failure process that defines the subpopulations discussed above. The two partition parameters of our model are the following points on the lifetime path:

- Partition parameter I/II at $t = 1$ when Phase I is passed and Phase II commences. Then subpopulation I, if present, is discarded. In our model one unit of time is defined as the duration of Phase I;
- Partition parameter II/III at $t = a$ when Phase II is passed and Phase III commences. Then subpopulations I and II are discarded whereas subpopulation III has survived up to $t = a$. From that point in time or service onwards the discarding process is mainly governed by economic aging and technical wear and tear. Obviously, the aging and wear and tear process will start from $t = 0$ onwards, however, it may have no consequences for discarding if early failures and/or sudden failures are decisive up to $t = a$. Discarding due to sudden failures applies also to subpopulation III after passing $t = a$. The distribution of lifetimes, $0 \leq t < \infty$, related to subpopulation III is termed <u>core distribution</u>.

The mathematics of this model will be elaborated in the following sections of this chapter.

III.4. The Three-component (Composite) Distribution

As we argued in Chapter II. above, it is reasonable to assume a WEIBULL-distributed service life. In this chapter the WEIBULL concept is taken as a working hypothesis for the elaboration of a suitable 3-component (composite) distribution model. The fit of this model to empirical retirement data is tested in Chapter IV and thereafter.

Our concept is constituted on the basis of three distinctive p.d.f.'s:

Phase I : $f_1(t)$ with parameters μ_1, β_1 for $0 < t \leq 1$

Phase II : $f_2(t)$ with parameters μ_2, β_2 for $1 < t \leq a$

Phase III: $f_3(t)$ with parameters μ_3, β_3 for $a < t < \infty$

as all p.d.f.'s start from t = 0.

Because the area under the probability density curve is one, it follows that:

$$\int_0^1 f_1(t|\mu_1,\beta_1)dt + \int_1^a f_2(t|\mu_2,\beta_2)dt + \int_a^\infty f_3(t|\mu_3,\beta_3)dt = 1 \qquad (1)$$

or, in short notation:

$$\{F_1(1) - F_1(0)\} + \{F_2(a) - F_2(1)\} + \{F_3(\infty) - F_3(a)\} = 1$$

Equation (1) is correct under the condition that:

$F_1(0) = 0$; $F_1(1) = F_2(1)$; $F_2(a) = F_3(a)$ and $F_3(\infty) = 1$

If a two-parameter WEIBULL distribution is assumed for each of the three phases, the universal p.d.f.'s are:

$$f_i(t) = \frac{\beta_i}{\mu_i} \left(\frac{t}{\mu_i}\right)^{\beta_i - 1} .\exp\left[-\left(\frac{t}{\mu_i}\right)^{\beta_i}\right] \qquad \text{for: } \begin{array}{ll} 0 < t \leq 1 & \text{and } i = 1 \\ 1 < t \leq a & \text{and } i = 2 \\ a < t & \text{and } i = 3 \end{array} \qquad (II/15)$$

and the survival function, as given above in Section II.4.2., is:

$$S_i(t) = \exp\left[-\left(\frac{t}{\mu_i}\right)^{\beta_i}\right] \qquad \text{for } i = 1,\ 2 \text{ and } 3 \qquad (II/17)$$

having 7 parameters: μ_1,β_1; μ_2,β_2; μ_3,β_3 and a, with a number of restrictions which allow for further simplification of the concept, as we will see in the next subsection.

III.4.1. Parametric Relationships in the Model.

We will now apply the following 3 restrictions:

1) $S_1(1) = S_2(1)$ for obvious reason;

2) $S_2(a) = S_3(a)$ for obvious reason;

3) $\beta_2 = 1$ because Phase II needs to have exponentially distributed T's

According to the fundamental starting points, the probability of survival at the end of each phase is equal to that at the beginning of the next phase, which implies 3 restrictions for the two partitions as defined before:

<u>Phase I/II:</u>

$$\exp\left[-\left(\frac{1}{\mu_1}\right)^{\beta_1}\right] = \exp\left[-\left(\frac{1}{\mu_2}\right)^{\beta_2}\right] \qquad \text{for } t = 1$$

From the above equation it follows that:

$$\mu_1^{\beta_1} = \mu_2 \qquad \text{for } 0 < \beta_1 < 1 \text{ ,and thus: } \mu_1 > \mu_2 \quad (2)$$

<u>Phase II/III:</u>

$$\exp\left[-\left(\frac{a}{\mu_2}\right)\right] = \exp\left[-\left(\frac{a}{\mu_3}\right)^{\beta_3}\right] \qquad \text{for } t = a \text{ and } \beta_2 = 1$$

Then it follows that: $\mu_2 = \mu_3^{\beta_3}/a^{\beta_3-1}$ for $\beta_3 > 1$ and $1 < a < \mu_3$ (3)

From these two sets of equations the following relationships between parameters are obtained:

$$\mu_1^{\beta_1} = \mu_2 = a^{(1-\beta_3)}.\mu_3^{\beta_3} \qquad \text{for } \beta_2 = 1 \qquad (4)$$

Due to the 3 restrictions mentioned above, the number of parameters is reduced from 7 to 4. We will use a, β_3, μ_3 and β_1.

As stated in Section II.6. above and elsewhere, more often than not $\beta_3 > 2$, because a linearly increasing hazard rate must be regarded as a limiting case for a progressively increasing aging and wear and tear. If $1 < \beta < 2$, the hazard rate would increase with a decreasing rate which is inconceivable in the case under consideration. In Section II.6. it was proved that the mode of a WEIBULL distribution is identical to the characteristic lifetime t* when $\beta = 2$. Since t* is defined as the moment at which the average consumption attains its minimum, we may conclude that $\beta = 2$, (a RAYLEIGH distribution) is a lower bound for β_3. This conclusion will be confirmed later and in Chapter IV when our concept is tested in the light of empirical retirement data.

According to the definition, Phase I must always be experienced. However, there is a very low probability of discard during that period. Bearing in mind that the probability of survival at $t = \mu$ is always $\exp[-1] = 0.368$, regardless of shape parameter β of the WEIBULL distribution, we are able to define a RAYLEIGH distribution for the limiting case of Phase III when $a = 1$ and Phase II is absent:

size parameter $\mu_0 = \mu_3$

shape parameter $\beta_0 = 2$

conceptual restriction $S_0(1) = S_1(1)$ for $a = 1$

In the limiting case for Phase III it follows that:

$$\exp\left[-\left(\frac{1}{\mu_1}\right)^{\beta_1}\right] = \exp\left[-\left(\frac{1}{\mu_0}\right)^{2}\right] \qquad \text{,and thus:}$$

$$\mu_1^{\beta_1} = \mu_0^{2} = \mu_3^{2} \qquad \text{for: } \begin{array}{l} a = 1 \\ \beta_0 = \beta_3 = 2 \\ 0 < \beta_1 < 1 \\ \mu_1 > \mu_0 = \mu_3 > 0 \end{array} \tag{5}$$

The duration of the "pre-aging" period (0,a), identical to Phase I for $a = 1$, and to Phases I + II for $a > 1$, can be derived from equation (4):

$$a = \left(\frac{\mu_3^{\beta_3}}{\mu_2}\right)^{1/(\beta_3-1)} = \left(\frac{\mu_3^{\beta_3}}{\mu_1^{\beta_1}}\right)^{1/(\beta_3-1)} \qquad \text{for: } \begin{array}{l} \mu_1 > \mu_2 > \mu_3 > 0 \\ 0 < \beta_1 < 1 \\ \beta_3 > 2 \end{array} \tag{6}$$

from which β_3 can be determined as follows:

$$\beta_3 = \ln(a/\mu_2)/\ln(a/\mu_3) \qquad \text{for } a > 1 \text{ and } \mu_2 > \mu_3 > 0 \tag{7}$$

Substituting (5) into (6) gives:

$$a = \mu_3^{\left(\frac{\beta_3-2}{\beta_3-1}\right)} \qquad \text{for } \beta_3 > \beta_0 = 2 \quad \text{,and:} \quad \mu_3 = \mu_2^{\frac{1}{2}} = \mu_0 > a > 1 \tag{8}$$

The probability of survival at time a must be higher than at the characteristic lifetime t*. Thus:

$$S_W(a) = \exp\left[-\left(\frac{a}{\mu_3}\right)^{\beta_3}\right] \geq \exp[-(1/\beta_3)] = S_W(t^*) \qquad \text{for } \beta_3 > 2 \tag{9}$$

This implies that: $(a/\mu_3)^{\beta_3} \leq 1/\beta_3$,and thus: $a \leq \mu_3.\beta_3^{-1/\beta_3} \leq t^*$

which is, indeed, in accordance with formula (II/30).

We have now shown all the parametric relationships and restrictions of our lifetime distribution model. We call the WEIBULL distribution with parameters (μ_3, β_3) the core distribution of our model.

III.4.2. Graphical Description of the modelled Distribution

Probability density function (1) of our 3-component (composite) WEIBULL distribution model as defined above is represented in Figure 3 below.

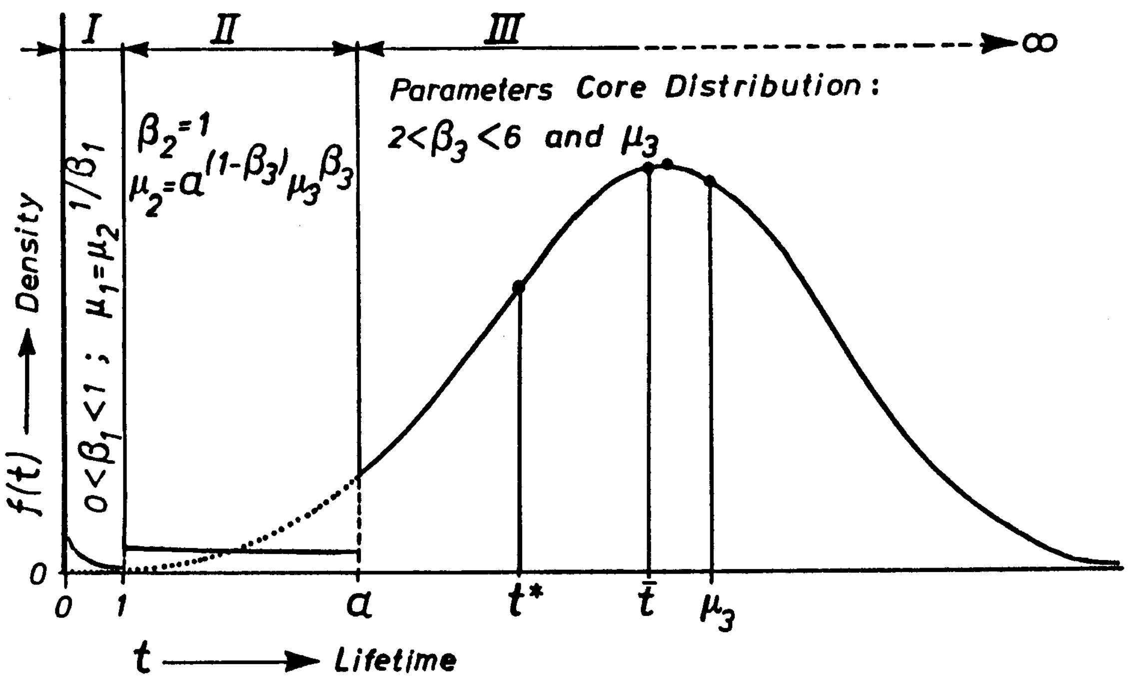


Fig. 3: Probability density curves of a 3-component (composite) WEIBULL distribution model.

The discontinuity in density at t = 1 and t = a coincides with the principles of the three distinctive risk-specific components of the (composite) distribution model. However, a fundamental characteristic of this model is that the integrated hazard curve is continuous and, for that reason, the cumulative distribution curve is continuous and S-shaped.

Since the model discussed above is related to its hazard characteristics, that subject is emphasized in Section III.4.3. in which the discontinuous hazard-rate curve as well as the continuous integrated hazard curve are discussed.

III.4.3. Three-component Hazard Concept

As already stated above, a universal lifetime distribution model may be developed from a 3-component (composite) model akin to the three successive and distinctive life phases characterized by decreasing, constant and increasing hazard rates. Such a hazard-rate pattern may represented by the "bathtub curve" familiar in maintenance and reliability engineering. However, our concept differs from the engineering one, as will be become apparent hereafter. It is stressed again that our concept has its rationale in the idea of risk-specific hazard functions as a reflection of the underlying process that generates a composite distribution. Figure 4 below represents our bathtub-shaped hazard-rate curves for the three successive and distinct life phases. According to the principles of our concept the hazard-rate plot is a 3-component curve with $h_1(0) = \infty$, $h_2(0) = 1/\mu_2$ and $h_3(0) = 0$. The pattern is not or not always continuous but presents two steps at t = 1 and t = a

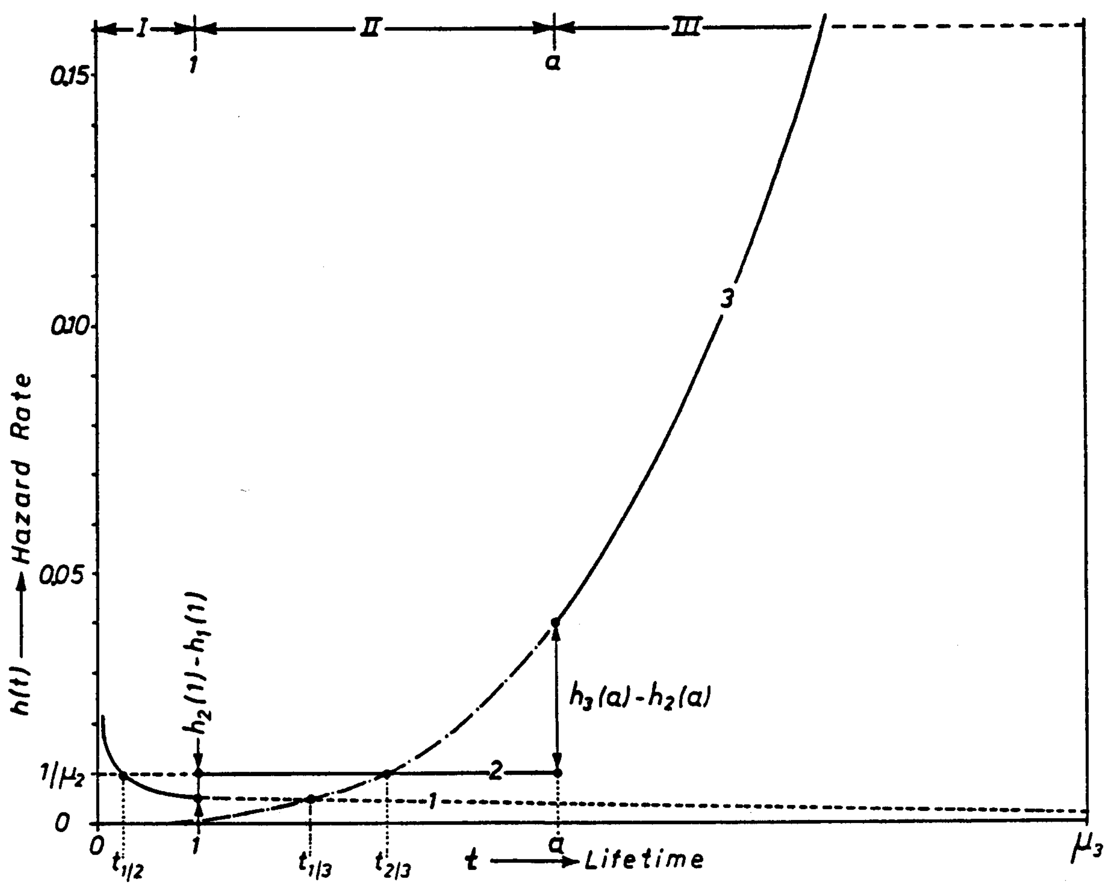


Fig. 4: Three-component hazard-rate plot for three successive and distinctive life Phases I, II and III.

where the distinctive lifetime distributions of the relevant subpopulations are truncated. A continuous monotonically increasing hazard-rate pattern is obtained when Phase I and II and, consequently, subpopulation I and II are absent. In the latter case there is only a single risk-specific hazardous process that governs the core distribution of the lifetime variable. Figure 4 illustrates clearly that our 3-component hazard-rate model differs from the bathtub-shaped plots which are frequently discussed in the literature on reliability theory. See, for instance, ASCHER and FEINGOLD (1984).

The distinctive hazard-rate curves 1, 2 and 3 generate WEIBULL distributions with parameters respectively:

Phase I : μ_1, $0 < \beta_1 < 1$

Phase II : μ_2, $\beta_2 = 1$

Phase III : μ_3, $\beta_3 > 1$, and probably, $\beta_3 > 2$.

If Phase I is applicable, curve (1) asymptotically decreases to zero. This means that the "learning/debugging/burn-in/running-in" process never stops. There is a continuous improvement possible but the effect on existing assets in service decreases with time and is overruled by either sudden change failures (curve 2) and/or disruptions caused by economic aging and technical wear and tear (curve 3).

In the real-world situation the partitions at t = 1 and t = a will not be as sharply focussed as illustrated by Figure 4 whereas the hazard-rate steps are smoothed out to some degree. Furthermore, the size of the hazard-rate steps are not as significant as suggested by Figure 4 which is calculated below on the bases of a WEIBULL model.

Hazard-rate step at t = 1:

$$h_2(1) - h_1(1) = (1 - \beta_1)/\mu_3^2 \quad \text{for } \mu_1^{\beta_1} = \mu_2 = \mu_3^2 > 0 \quad \text{and } 0 < \beta_1 < 1 \qquad (10)$$

Hazard-rate step at t = a when a is defined by (8):

$$h_3(a) - h_2(a) = (\beta_3 - 1)/\mu_3^2 \qquad \text{for } \mu_2 = \mu_3^2 > 0 \quad \text{and } \beta_3 \geq 2 \qquad (11)$$

The quantities according to formulae (10) and (11) are very small. As an example, if μ_3 = 10 years; β_1 = 0.5 and β_3 = 3, it follows that:

$h_2(1) - h_1(1) = 0.005$ p.a. ,and: $h_3(a) - h_2(a) = 0.02$ p.a.

Formula (11) suggests that the associated hazard-rate step is independent of a. However, a is according to (8) related to the parameters of the core distribution.

Figure 4 shows that the intercept $h_2(1)$ is also reached at the crossings of h_1 and h_2, and of h_2 and h_3, i.e. at:

$$t = t_{1/2} = \beta_1^{\,1/(1 - \beta_1)} \qquad \text{for } 0 < \beta_1 < 1 \tag{13}$$

$$t = t_{2/3} = \left(\frac{\mu_3^{\beta_3 - 2}}{\beta_3} \right)^{1/(\beta_3 - 1)} \qquad \text{for } \beta_3 > 2 \tag{14}$$

The value $h_2(1)$ is also reached at the crossing of h_3 and h_1, i.e. at:

$$t = t_{1/3} = \left(\frac{\mu_3^{\beta_3 - 2}}{\beta_3/\beta_1} \right)^{1/(\beta_3 - \beta_1)} \qquad \text{for } \beta_3 > 2 \tag{15}$$

Furthermore:

$t_{1/2} < t_{1/3} < t_{2/3}$; $0 < t_{1/2} < 1$ and $1 < t_{2/3} < a$.

The fundamental characteristic of a 3-component hazard-rate function is that its integrated hazard function is represented by an unbroken pattern as can be derived from Section III.4.1. The integrated hazard at the first partition, t = 1, is according to formula (II/28) and equation (2):

$$H_I(1) = \left(\frac{1}{\mu_1}\right)^{\beta_1} = \frac{1}{\mu_2} = H_{II}(1) \tag{16}$$

The same applies for the integrated hazard on the basis of equation (3) at t = a (second partition):

$$H_{II}(a) = \frac{a}{\mu_2} = \left(\frac{a}{\mu_3}\right)^{\beta_3} = H_{III}(a) \tag{17}$$

The integrated hazard pattern differs for the three successive and distinctive components due to different risk-specific hazard rates but it is continuous in the domain (0,t), thus also in the points t = 1 and t = a.

The integrated hazard curve according to our model is illustrated by Figure 5 below.

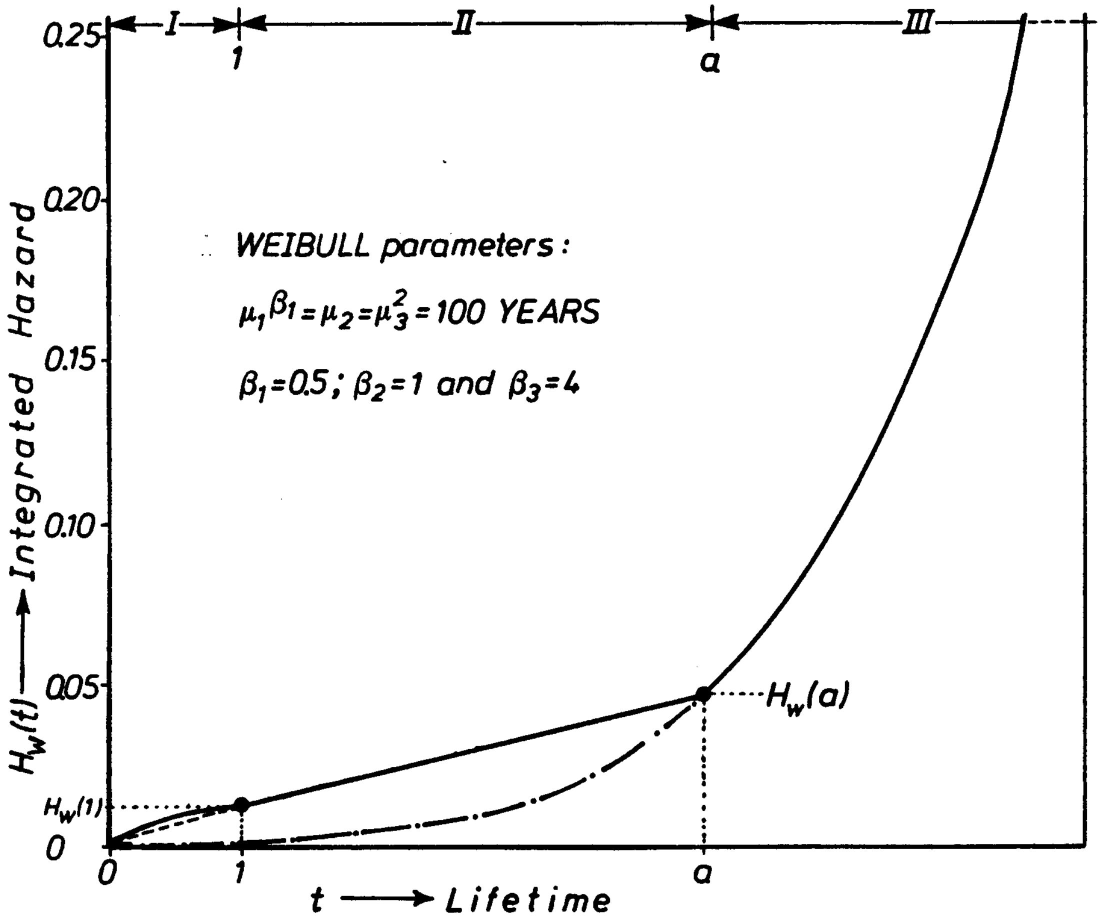


Fig. 5: Three-component integrated hazard plot for three successive and distinctive life Phases I, II and III.

Herewith the 3-component risk-specific hazard-rate and integrated-hazard concept are mathematically defined and graphically presented.

III.5. Characteristic Shape Parameters

In Section II.6. the characteristic lifetime, t*, was formulated (II/30) and defined as the point in time at which the average capital consumption has attained its minimum value. For the core WEIBULL distribution, t* may be written as follows:

$$t^* = \mu_3.\beta_3^{-1/\beta_3} \qquad \text{for } \mu = \mu_3 \text{ and } \beta = \beta_3 \geq 2 \tag{18}$$

Furthermore, the "average capital consumption" was formulated (II/9) as I/T.S(T). When T = t* and $S(T) = S_W(t^*) = \exp[-(t^*/\mu_3)^{\beta_3}]$ are inserted in (II/9), the minimum "average capital consumption", defined by (II/5), becomes:

$$Y_C(t^*) = (I/\mu_3).\beta_3^{1/\beta_3}.\exp[1/\beta_3] = m(t^*) \tag{19}$$

where (I/μ_3) is the average depletion of I over time interval $(0,\mu_3)$. In view of depreciation and replacement (see Chapters V and VI), a realistic option is that $2(I/\mu_3) = m(t^*)$:

$$Y_C(t^*) = 2.(I/\mu_3) \quad \text{,and thus:} \quad \beta_3^{1/\beta_3}.\exp[1/\beta_3] = 2 \tag{20}$$

From (20) follows a "characteristic shape parameter" of the first kind written as $\beta_3 = \overset{*}{\beta}_3[1] = 3.05$.

On a closer examination of integrated hazard function (II/28), associated with a WEIBULL distribution of lifetimes, it is striking that size parameter μ is the point of inflexion. This is also demonstrated by the WEIBULL elasticity function (II/27) which embodies (II/28). When t = μ, the elasticity $\pi_W(\mu) = 0$ independent of β as in the case of the integrated hazard at t = μ. The break point in time μ is clearly expressed by taking the logarithm of the WEIBULL integrated hazard function:

$$\ln H_W(t) = \ln (t/\mu)^{\beta} = \beta.\ln t - \beta.\ln \mu \tag{21}$$

In a plot with x = ln t and y = ln H(t) straight lines are obtained for all β's which have a common intercept on the x-axis with coordinates x = ln μ and y = 0. This fan of straight lines has another remarkable property at t = t* when the average capital consumption is reduced to a minimum. Then, in accordance with formula (II/31), the following applies for $\beta = \beta_3$:

$$H_w(t^*) = 1/\beta_3 \quad \text{,and thus:} \quad y = \ln H_w(t^*) = -\ln \beta_3 \tag{22}$$

When (21) is represented by a straight line with slope β through the origin, $x = \ln \mu_3$ and $y = \ln H_w(\mu_3) = 0$, two characteristic points ($t^* < \mu_3$ and $t_s^* > \mu_3$) symmetrically positioned on that line on either side of the origin can be described by the following set of equations:

X-direction: $\ln \mu_3 - \ln t^* = \ln t_s^* - \ln \mu_3$

Y-direction: $-\ln H_w(t^*) = \ln H_w(t_s^*)$ (23)

Since according to (22), $H_w(t^*) = 1/\beta_3$, it follows from (23) that:

$$H_w(t_s^*) = \beta \tag{24}$$

where t_s^* is the symmetrical characteristic lifetime.

Next, the parts of the core distribution from t^* to μ_3 and from μ_3 to t_s^* are considered. When the areas under the p.d.f.-curve on the left of μ_3 to t^* and on the right of μ_3 to t_s^* are equal, it follows that:

$$S_w(t^*) - S_w(\mu_3) = S_w(\mu_3) - S_w(t_s^*) \tag{25}$$

which gives:

$$2.S_w(\mu_3) = S_w(t^*) - S_w(t_s^*) = \exp[-H_w(t^*)] - \exp[-H_w(t_s^*)]$$

After substituting $S_w(\mu_3) = \exp[-1]$, $H_w(t^*) = 1/\beta_3$ and $H_w(t_s^*) = \beta_3$, it follows that:

$$2.\exp[-1] = \exp[-(1/\beta_3)] - \exp[-\beta_3] = 0.735759$$

From the above equation we obtain:

$$\beta_3 = \beta_3^*[2] = 3.667 \tag{26}$$

which is the characteristic shape parameter of the second kind.

A further examination shows that the probability of survival at $t = t_s^*$ is very low. For $\beta_3^* = 3.667$, for instance, $S_w(t_s^*) = 0.025553$; for $\beta_3^* = 4.6$, the probability of survival at $t = t_s^*$ decreases to 0.01.

There is another measure for a very low probability of survival when the hazard rate is one. Then, for a WEIBULL hazard rate it applies that:

$$h(t_1) = \left(\frac{\beta_3}{\mu_3{}^{\beta_3}}\right).t_1^{\beta_3 - 1} = 1 \tag{27}$$

where t_1 is the lifetime which just satisfies the hazard-rate restriction. From (27) it follows that:

$$t_1 = \mu_3^{\beta_3/(\beta_3 - 1)} . \beta^{-1/(\beta_3 - 1)} \tag{28}$$

Since $t^* = \mu_3 . \beta_3^{-1/\beta_3}$, it follows from (22) and (24) that:

$$t_s^* = \mu_3 . \beta_3^{1/\beta_3} \tag{29}$$

Combining (28) and (29) gives:

$$t_1 > t_s^* > \mu_3 \qquad \text{for } \beta_3^{1/\beta_3} > 1$$

This result implies that the hazard-rate restriction does not apply to the characteristic lifetime of the second kind.

In addition to the first and second characteristic lifetimes, a third one can be derived. For this case the part of the core distribution from t^* to μ_3 is considered again. With μ_3 as the break point in time, the other part considered is from μ_3 to infinity. Assuming again that the areas under the p.d.f.-curve between t^* and μ_3, and on the right of μ_3 to ∞ are equal, it follows for a WEIBULL distribution that:

$$S_W(t^*) - S_W(\mu_3) = S_W(\mu_3) \tag{30}$$

which proceeds for a WEIBULL distribution to:

$$2.S_W(\mu_3) = S_W(t^*)$$

Since $S_W(\mu_3) = \exp[-1]$ and $S_W(t^*) = \exp[-H_W(t^*)]$, it follows that:

$$2.\exp\left[-1\right] = \exp\left[-(1/\beta_3)\right]$$

From the above equation we obtain:

$$\beta_3 = \beta_3^*[3] = 1/(1 - \ln 2) = 3.2589 \tag{31}$$

which is the characteristic shape parameter of the third kind. See also (V/28) in Chapter V.

It can be shown that in the latter case the "mode" and the "median" of a WEIBULL distribution are identical. According to formulae (II/23) for the mode and (II/24) for the median, the following equality is obtained:

$$\hat{t}_W = \mu\{1 - (1/\beta)\}^{1/\beta} = \mu\{\ln 2\}^{1/\beta} = \mathring{t}_W$$

It follows that $1 - (1/\beta) = \ln 2$ and, consequently, $\beta = 3.2589$. When "mode" and "median" are identical, the WEIBULL distribution is approximately symmetrical.

Now we have derived the following three characteristic shape parameters:

$$\beta_3^*[1] = 3.05 \qquad \beta_3^*[2] = 3.667 \qquad \beta_3^*[3] = 3.2589$$

In fact, WEIBULL distributions with these three characteristic shape parameters are all roughly symmetrical. The one with $\beta_3^*[1]$ is slightly left modal; the one with $\beta_3^*[2]$ is slightly right modal. The above findings imply that other symmetrical distributions, such as a NORMAL distribution and a LOGISTIC distribution also may fit for Phase III alone. A great proportion of the WINFREY-curves employed in practice are also symmetrical. We return to this subject in Chapter IV.

If the quantity (β_3^{-1/β_3}) according to Figure 2 (Section II.6.) is almost constant for a wide range of shape parameters, so is its reciprocal (β_3^{1/β_3}) in the ratio $t_S^*/\mu_3 \approx 1.43$. Note that this is not necessarily true in every case of a reciprocal.

When (18) and (29) are combined, we have:

$$t^*.t_S^* = \mu_3^{\,2} = \mu_2 \qquad (32)$$

This again makes it clear how important the size parameter of the core distribution is, just as the shape parameter is relatively unimportant between $1.6 < \beta_3 < 6$. However, there is a significant (optimum) characteristic range of $3.05 < \beta_3^* < 3.67$.

These findings are essential for modelling and implementation in practice as well as for theoretical work on the subject of lifetime distribution concepts.

III.6. Graphical Form of the Model

The amount of capital consumed, $C(T)$, in time interval $(0,T)$ is a starting-point which is discussed in Chapter II. According to (II/8), that quantity is $C(T) = I/S(T)$.

Converted to its logarithmic form, we became:

$$-\ln S(T) = \ln C(T) - \ln I = H(T) \tag{II/37}$$

Function (II/37) shows a direct relationship between the integrated hazard and the important ratio $C(T)/I$ in time interval $(0,T)$. This subject is comprehensively discussed in Chapter V.

In Section III.5. we have demonstrated that the log integrated hazard rate function (21) of our model is linear. In a plot,

$$x = \ln t \quad \text{,versus} \quad y = \ln.\ln\{1/S_w(t)\} = \ln H_w(t),$$

we then obtain straight lines with slope β. Figure 6 on the next page illustrates our concept for Phase I, II and III. Phase I is partly exposed starting arbitrarily from $\ln t = \ln 0.5$; this phase is disregarded in the next explanation.

All triangles in Figure 6 are determined by the three encircled points with the following coordinates:

1. Origin: $x = \ln t = 0$ when $t = 1$, and $y = \ln H_w(1)$
2. Intercept of survivor curve II and III:
   $x = \ln a$ when $t = a$, and $y = \ln H_w(a)$
3. On the top: $x = \ln \mu_3$ when $t = \mu_3$ and $y = \ln H_w(\mu_3) = 0$

Since $\ln H_w(a) = \beta_2.\ln a/\mu_2 = \beta_3.\ln a/\mu_3$, four data must be known in order to construct the triangle concerned. According to the restricions of our model the crucial data are: a, μ_3, β_3 and $\beta_2 = 1$.

From examination of the triangle, it is clear that once $H_w(a)$ and a are fixed, all remaining parameters of our model can be determined, whereas the size parameter, $\mu_0 = \mu_3$, of the core distribution can be derived from a fixed $H_w(1)$ alone. The latter is an essential finding which is confirmed both in science and in practice. As an example, a human's initial vitality largely determines his/her probability of survival to a certain age, say seventy years, barring accidents. Another example is the strength of a material or construction; crack propagation under cyclic stress depends on the initial crack length determining the initial condition, i.e., $H(1)$.

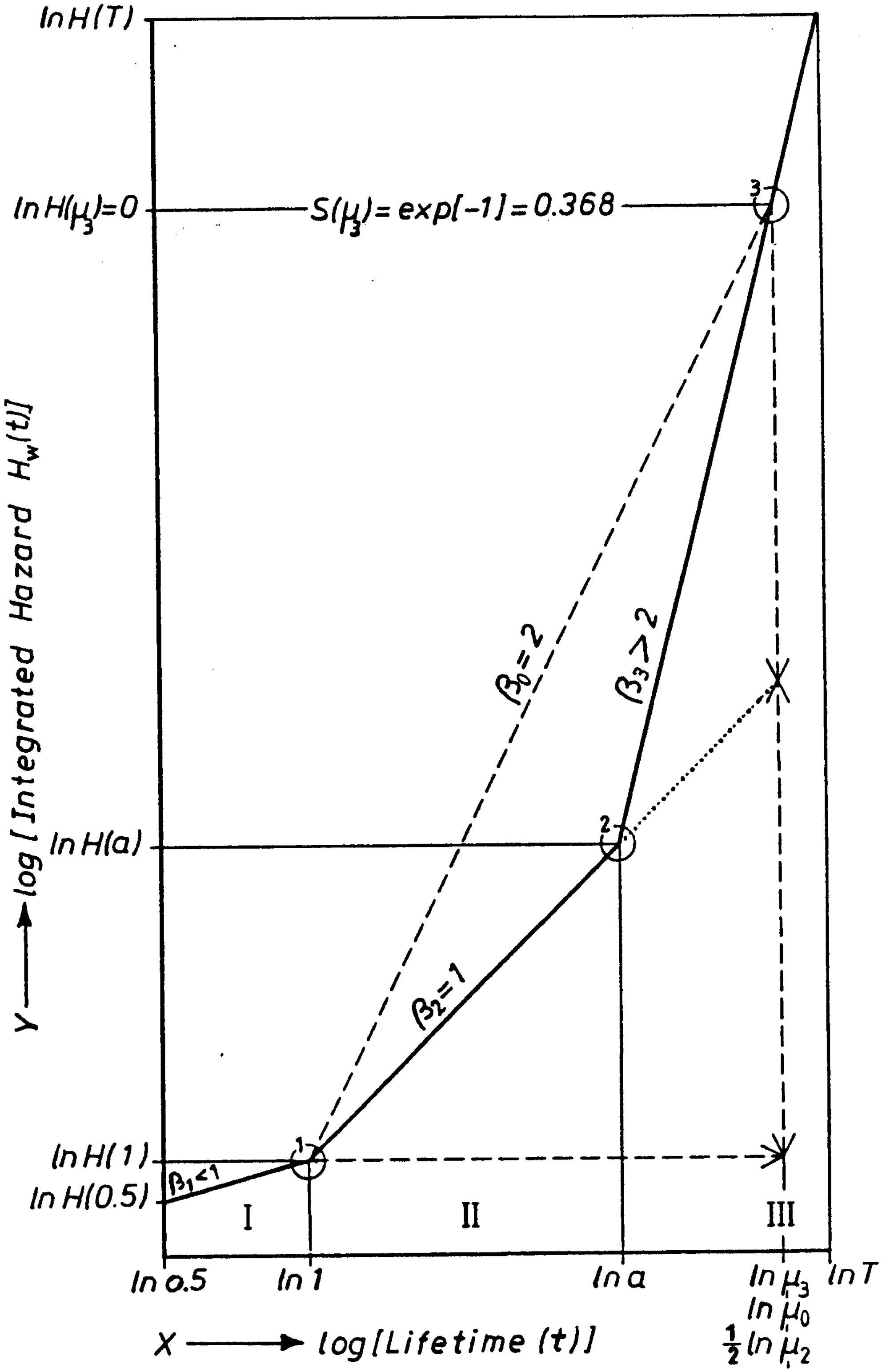


Fig. 6: Linear plot of ln $H_w(t)$ versus ln t representing a triangle-based lifetime distribution model of capital assets and manufactured durables.

The validity of this for capital assets such as production machinery and equipment is also evident; the initial degree of (surplus) fitness is the main consideration for life expectancy. A simple and significant example is a vehicle tyre. Apart from the quality of the rubber and the structure, the initial height of its profile determines its lifespan, barring accidents. In fact, a new tyre initially has a surplus fitness for purpose if its profile height is 10 mm whereas 2 mm is the lowest limit for the purpose. The initial condition and associated vitality of productive capital assets and manufactured durables in service is generally expressed by their potential surplus performance or their potential surplus responsive capacity to life-attacking forces (shocks), which is reflected by the origin at $t = 1$ and $H_w(1)$.

The graphical represention according to Figure 6 will be applied in Chapter IV to analyse retirement data.

CHAPTER IV

TESTING OF PROBABILISTIC LIFETIME MODEL

IV.1. Introduction

The purpose of this chapter is to test the validity of our probabilistic lifetime model and its fundamental principles as developed in Chapter III. Testing would not be difficult if there were sufficient and appropriate retirement data which were accurate and fully documented. If this was the case, there would be less need for a lifetime model as an approximation of the real world and as an instrument to be used by scientists and professionals who deal with the life characteristics of capital assets and manufactured durables. WINFREY (1931/1935) was also faced with the problem of insufficient, inappropriate and poorly documented empirical retirement data. For the same reason he decided to develop curves which give tabulated survival percentages versus age (in percent of average age). He dealt with the effects of discarding, not with the causes, i.e., not with the underlying discarding process. He used incomplete and inaccurate empirical data to construct survivor curves which is typical in an effect-based approach. In this way the validity of the constructed survivor curves can not be demonstrated.
Our theoretical model is based on WEIBULL distributions generated by some hazardous life-attacking mechanism. Discarding is caused by an underlying hazardous process specified for the model to be tested. Consequently, a causality-based approach of testing can be employed. Then the use of incomplete, inaccurate and poorly documented empirical retirement data for testing purposes brings fewer difficulties because testing can be combined with data diagnostic techniques. The latter may, for example, reveal the effect of different hazard-specific implications and/or a number of discrepant observations.

It is always comforting to be able to justify a plot of data points that describes a random lifetime variable on more than just empirical retirement data. An appropriate plot may indicate ways of remedying misspecifications whereas it is often useful to be able to interpret parameters in probability distributions in terms of the characteristics of the underlying process that generates the random variable. Since our lifetime distribution model is represented by a linear (triangular) plot related to a composite family of WEIBULL distributions, it is natural to use a preliminary graphical analyses followed by analytical techniques for estimating relevant parameters. In this way the model offers a facility for data diagnostic techniques through preliminary and informal graphical analyses and, simultaneously, the model itself is tested by the applications of refined methods in estimating its parameters.

In Section IV.2., 96 sets of empirical retirement data are discussed which are derived from the following 4 main sources:

1. Engineering-works Machinery and Equipment, consisting of mechanically operated tools for machining of metal workpieces in one and the same large engineering works.
2. Dalcy (Database Lifetime Cycle), enclosing more than 36,000 records from which 20 sets (2,275 discards) are pre-selected for testing purposes.
3. WINFREY original empirical retirement data documented in BULLETIN 103 (1931) and BULLETIN 125 (1935).
4. Miscellaneous sets consisting of dwellings documented by BEKKER (1980), passenger cars and bus tyres. The latter kind of goods are considered as simplified capital assets with similar life characteristics.

Section IV.3. is devoted to the problem of aggregated sets of empirical retirement data, and the problem of erroneous data. The impact of aggregation and of measuring and recording errors on the value of parameters is investigated by means of simulation techniques. The results are discussed in Section IV.3.1..

The testing procedures are described in Section IV.4. starting with the nonparametric estimate of the survivor function as developed by KAPLAN & MEIER (1958). Section IV.4.1. deals with preliminary graphical techniques which are applied for segregation of data points related to Phase I, II or III. Then a quasi-linear regression technique is considered in Section IV.4.2. and employed for parameter estimation. In Section IV.4.3. the maximum-likelihood approach to estimation is discussed. As a check on how far the hypothetical survival functions and their parameter estimates correspond with empirical findings, analyses of discrepancies are carried out. The residual analyses for the integrated hazard are discussed in Section IV.4.4.. The method is a graphical technique by which peculiar data as well as misspecifications, if any, can be seen. In addition to plotting residuals for the integrated hazard, discrepancies related to the probability of survival are calculated. This is a direct approach of model testing which is discussed in Section IV.4.5.. The tabulated testing results of 30 representative and selected sets are inserted in that section and discussed in Section IV.4.5.1.. These results are considered in more detail in Section IV.5. (data source 1), Section IV.6. (data source 2), Section IV.7. (data source 3) and Section IV.8. (data source 4).

Finally, in Section IV.9. the testing experience and results are considered leading to a conclusion with respect to the validity of the lifetime model concerned in this study.

IV.2. Explored Empirical Retirement Data

Empirical retirement data and life-characteristic information on reproducible and depreciable capital assets as well as on manufactured durables are very sparse. In this study we failed to find any completely measured and described sets of empirical retirement data. Consequently, it remained unknown whether they were (singly or multiply) censored or not as is commonly known in engineering and medical life testing practice. Nevertheless, many valuable sets of empirical retirement data are explored including a great deal of the original WINFREY data. Each set will be analysed and discussed in more detail in a separate section.

Altogether 96 sets of empirical retirement data are employed. They are derived from the following 4 main sources:

1. Engineering-works Machinery and Equipment

   Source: Central Bureau of Statistics, Heerlen, Netherlands (1982).

   The retirement data are concerned with milling equipment, lathes, drilling equipment, grinding equipment, welding equipment, surface-treating equipment, squeezing, punching, drawing and stretching equipment which have been in use as mechanically operated tools for machining of metal workpieces in one and the same large engineering works. The lifetimes of individuals are measured and recorded in years (age classwidth: one year).

   This engineering works used in their administration 4 age classes for mechanically operated tools (equipment): 7, 10, 14 and 20 years. The equipment considered operated partly in one and partly in two shifts. The age class for a two-shift operation is, generally, one class higher than the class of a one-shift operation. The record includes 424 units which were discarded in the period 1976-1979. A great deal of the equipment withdrawn was not scrapped at the moment of discarding but transformed via the second-hand market to a lower class of work (function degradation).

   The required condition of these mechanically operated tools was maintained by repairs. The condition was restored to "as bad as old" corresponding to the initial condition just after purchasing. Of course, each time it was decided whether repairs or replacement should

be the most economic policy with respect to the productive service function of the equipment considered. See NEWBY (1987).
In fact the records are concerned with aggregates of a certain kind of equipment. For example, drilling includes a range of equipment from small hand-drilling units to large single and multi-column drilling equipment, and from manually to semi and fully automated controls. Others within a category are even more aggregated, e.g., surface-treating equipment consists of several kinds of mechanically operated tools which treat the surface of a metal workpiece in different ways. The same applies to welding, squeezing, punching, drawing and stretching equipment. None of the records is concerned with identical goods which start their productive life on the same date. However, their function within a certain kind of equipment is more or less the same. The function of all drilling equipment, e.g., is to make specified holes in a workpiece. The basic function of welding equipment is to connect metal parts or components which can be done by several kind of welding equipment designed for different purposes. Obviously, the aim of mechanically operated tools is to accomplish their productive function in the most economic manner. From that point of view a functional division of capital assets is a practicable idea. In the case under consideration some kinds of capital assets are divided into purchase price classes converted to constant prices (1975). In order to achieve a more homogeneous group, we have selected equipment of each kind according to its purchase price class. This was possible for lathes and for milling, drilling and grinding equipment as listed in Table IV-1, Appendix VII.1, page 1.
As can be derived from Table IV-1, this category of capital assets covers 8 sets and 313 discards of individuals which are selected for further analyses from 16 sets and 424 discards. Eight sets were rejected because of a lack of information or a too high degree of heterogeneity.

2. <u>Dalcy (Database Lifetime Cycle)</u>
   Source: Central Bureau of Statistics, Heerlen, Netherlands (1987).

This newly developed database of individuals contained in 1987 more than 36,000 records of capital assets related to the industrial sector. Each record contains among other information the year of purchase and of discarding, the economic activity of the asset

concerned, the size class of the industry where the equipment has been used, its category and its purchase price. Furthermore it is indicated whether the asset in question at the end of its service life is scrapped or transformed to another (lower) class of capital stock.
As in the foregoing group (Source 1), none of the records deals with identical goods which started their productive life on the same date, however, their function within a certain kind of capital is more or less the same. Nevertheless, the heterogeneity is significant. This can be reduced to a certain extent by selecting equipment with an identical function according to their purchase price class.
For the case under consideration we have chosen the following main groups and categories of which the lifetimes of individuals are measured and recorded in years (age classwidth: one year):

- Internal (mechanical) transportation equipment including the categories:
  * forktrucks
  * roller conveyors
  * continuous conveying equipment
- External (rolling) transportation equipment including the categories:
  * Passenger and delivery cars including combines (maximum weight, loaded, less than 5 tons)
  * Trucks, delivery cars, mobile cranes and mobile fire engines (maximum weight, loaded, more than 5 tons)
- Industrial equipment, apparatus and installations including the categories:
  * Wrapping equipment
  * Pumps and compressors
  * Electric generators
  * Welding and flame-cutting equipment
  * Measuring and controlling equipment
- Engineering works machinery and equipment including the categories:
  * Metal chipping tools (lathes and milling, drilling and grinding equipment, etc.)
  * Non-chipping tools (squeezing, punching, drawing and stretching, forging and bending equipment, etc.)

In fact, the mechanically operated tools mentioned above under Source 1 belong to the latter category.

Again, the categories are selected according to their purchase price classes (prices 1980) and listed in Table IV-2, Appendix VII.1., page 1.

As can be derived from Table IV-2, this category of capital assets covers 20 sets and 2,275 discards of individuals which are selected for further analyses. The remaining records were rejected because of irrelevance or a too high degree of heterogeneity.

3. WINFREY Retirement Data

Source: Bulletin 103, IOWA Engineering Experiment Station (1931).

This well-known report deals with "life characteristics of physical property" emphasized on the development of classes of survivor curves. WINFREY (1931) presented a method that he described as the calculation of a mortality curve, the probable life curve, and the rate of renewals of particular examples and types of physical equipment. The method was applied to 65 sets of original life data for property found in the following industries: water supply, telephone, telegraph, electric service, electric railway, steam railroad, agricultural implement, and motor vehicle. In 1931 WINFREY presented 13 type curves: 4 left-modal curves, 5 symmetrical curves and 4 right-modal curves.

Four years later Bulletin 125 was published wherein the number of empirical retirement data sets was increased from 65 to 176. In the latter report WINFREY (1935) presented 18 types survivor curves: 6 left-modal curves, 7 symmetrical curves and 5 right-modal curves. The property goods previously studied by WINFREY (1931) are listed in Table IV-3, Appendix VII.1, page 2, which is derived from Bulletin 125 (pp.142-144).

All 65 sets listed in Table IV-3 are used for checking and judging our lifetime model whereas the additional 111 sets studied by WINFREY (1935) and recorded in Bulletin 125 are left out of consideration because the original retirement data source got lost. As can be derived from the table, several property goods are of the same kind. Therefore, 15 representative sets are studied in more detail in Sections IV.7. and IV.7.1.. The sets deal with individual observations, however, the number of individuals in several sets is unknown. Lifetimes are recorded in years (age classwidth: one year).

4. Miscellaneous Sets

These empirical retirement data deal with:

- Dwellings in the Netherlands as investigated previously by BEKKER (1980).
- Passenger cars in the Netherlands as investigated by VOORDOUW (1981).
- Bus tyres which were used on 200 buses for public transport in the city of The Hague (Netherlands). This lifetime study was initiated by the Research Institute for Management Science at Delft (presently at Maastricht) and reported by TARIGAN (1985).

In the Netherlands the number of dwellings withdrawn from stock is registered yearly. Their age in the year of discarding is also registered. The service life analysis covered the period from 1800 to 1976. During that long period of 176 years this stock was gradually built up. BEKKER (1980) demonstrated that the number of a certain age class discarded annually is proportional to the number of dwellings built at that time. The previous investigation on the lifetime of dwellings produced 48 data points in a survivor plot as a result from 6 age classes and 8 observation years. For the study ahead the same 48 points are used and, in addition, the original data are re-arranged in such a way that a 12 points survivor plot can be constructed. The miscellaneous sets of empirical retirement data are listed in Table IV-4, Appendix VII.1, page 1. Additional information is given in Section IV.8..

The number of dwellings that was discarded from the dwelling stock in the Netherlands in the period 1961-1976 amounted to more than 12,500 units p.a.. Also the number of passenger cars discarded from the Netherlands stock each year is very high. Furthermore, the 1977 empirical retirement data of individuals are reliable to an age when the probability of survival has decreased to less than 5%.

From a statistical point of view, the bus tyres are very interesting. These tyres were mounted on the same type of buses which are randomly put on all public bus routes in one and the same city. Furthermore, the impact of a busdriver on the wear out of tyres is not in effect because all drivers work shifts and are allocated randomly to buses. Since these individual tyres can be retreaded or not, the replacement decision making process is an economic one as in the case of capital assets or manufactured durables in service. Therefore, a similar lifetime distribution model may be expected.

Here we have given an overview of the 4 tables of the empirical retirement data used for checking and judging our lifetime model. More details on this material and on each separate set will be discussed in this chapter later on in Sections IV.5. to IV.8..

IV.3. Effects of Aggregation and Measurement Errors

In Section IV.2. above it became clear that practically all sets of empirical retirement data are more or less aggregated but that the degree of aggregation is unknown. This problem could not be solved analytically. Therefore it was decided to investigate the impact of aggregation and of measuring and recording errors on the value of WEIBULL parameters using simulation.

We assume that a given set of empirical retirement data of individuals relates to a collection of different subpopulations of capital assets which belong to one category (e.g., mechanically operated tools in the engineering sector), that each subpopulation relates to a homogeneous mass of one kind of capital assets, and further, that the lifetime variable of each subpopulation is WEIBULL distributed, although the size parameter of each distribution differs in value and the shape parameters may be equal or not. With reference to the example above we may have subpopulations of milling, grinding, drilling, welding, surface-treating, shearing, punching and cutting equipment which can reasonably be regarded as homogeneous.

Apart from different size parameters of the distributions related to the different subpopulations, we have the problem of inaccurate measurement and recording of the point in time when an asset is discarded from a certain class of stock. If the life span is measured in years, as usually, the average error in the raw data will be half a year. Sometimes the service is measured in productive (running) hours, days, weeks or months and not in (integer) calendar years. It may be clear that a use-based lifespan can (will) differ from an age-based lifespan. Operating in one or more shifts may affect life characteristics of identical productive equipment in the industrial sector. Identical passenger cars of the same age will differ in terms of running kilometres and, consequently, in their probability of survival. In practice records are not converted to uniform units of lifetime. Therefore empirical retirement data are, generally speaking, raw and inaccurate.

The problem of aggregation may be solved by assuming a homogeneous mass of which empirical retirement data are erroneously measured and recorded. That is to say, the exact lifespan of each object is not measured and recorded but the erroneously measured lifespan is recorded. Then, we have the errors-in-variables problem.

To investigate the impact of measurement errors, we used a computer simulation following DIJKMEIJER (1990). This simulation program is based on the lifetime model of Chapter III. A hypothetical WEIBULL concept is employed with parameters (μ_3, β_3). When the empirical retirement data refer to a reasonably homogeneous population of capital assets or manufactured durables, and when measured sufficiently accurately, the assumed value of the shape parameter of the core WEIBULL distribution of lifetimes may range from $2 < \beta_3 < 6$. A value of $\beta_3 < 2$ is not expected under bona fide conditions whereas a value of $\beta_3 > 6$ is possible but unusual in common practice. Therefore, a hypothetical shape parameter $\beta_3 = 4$ was chosen and, alternatively, $\beta_3 = 2$ and $\beta_3 = 6$ as threshold values with respect to simulated inaccuracy.
Three values for the size parameter of the hypothetical WEIBULL distribution are chosen for three distinct simulation runs, namely $\mu_3 = 10$; $\mu_3 = 25$ and $\mu_3 = 50$ years. Of course, any value of μ_3 is possible but the chosen range is appropriate for the investigation of the impact of erroneous empirical retirement data due to (quasi) inaccurate measuring and recording.
The simulation starts with the partition of the lifetime scale into $(n - 1)$ equal periods. Then we have n partition points in time to denote by:

$$T_i \quad ,\text{with: } i = 1, 2, 3, \ldots.. n$$

where:

$$T_1 = \mu_3^{1-(2/\beta_3)} \quad ,\text{and} \quad T_n = \mu_3 . 15^{1/\beta_3}$$

These times correspond to the following probability of survival values:

$$S(T_1) = \exp[-(1/\mu_3^2)] = \exp[-(1/\mu_2)]$$

$$S(T_n) = \exp[-15] = 0.000000306$$

The number of partition points depends on the class size of lifetimes. For a class size of 0.1 units of time we have:

$$\frac{T_n - T_1}{n - 1} = 0.10 = \text{class size of lifetimes} \qquad (1)$$

The number of partition points n calculated by means of (1) for the three different shape and size parameters are given below:

μ_s = 10 years:	$\beta_s = 2$	$n =$ 378	partition points
	$\beta_s = 4$	$n =$ 166	
	$\beta_s = 6$	$n =$ 112	
μ_s = 25 years:	$\beta_s = 2$	$n =$ 959	partition points
	$\beta_s = 4$	$n =$ 443	
	$\beta_s = 6$	$n =$ 308	
μ_s = 50 years	$\beta_s = 2$	$n =$ 1,928	partition points
	$\beta_s = 4$	$n =$ 914	
	$\beta_s = 6$	$n =$ 651	

Now every T_i of each of the hypothetical WEIBULL distributions with 3 distinctive sets of chosen parameters is known. Then the related probability of survival, $S(T_i)$, can easily be calculated.

Since T_i is erroneously measured as $T_{i.\Phi}$, it is assumed that the values of $T_{i.\Phi}$ are symmetrically distributed with mean T_i. The variance, $\sigma^2_{i.\Phi}$, of that distribution is assumed to be proportional to T_i; in other words, the coefficient of variation rather than the variance was taken as a constant. Furthermore, it was assumed that the erroneously measured retirement data are roughly NORMAL distributed. Instead, a truncated distribution was taken which is numerically more convenient to the simulation program. To approach a NORMAL distribution and to avoid (a very little change on) negative values, a CAUCHY distribution was used. Its long tails were vertically truncated on both sides.

The generated (simulated) value of $T_{i.\Phi}$ is paired with $S(T_i)$. Then we have:

$$S(T_i) = S(T_{i.\Phi}) \tag{2}$$

This is only possible if $\mu_{i.\Phi}$ and/or $\beta_{i.\Phi}$ differ from parameters μ and β of the hypothetical WEIBULL distribution. This is in agreement with what we wish to investigate, namely the effect on the parameter estimates of the hypothetical WEIBULL distribution due to measurement errors. The estimates are:

β_{Φ} = shape parameter estimate of the distribution of the simulated data points

μ_{Φ} = size parameter estimate of the distribution of the simulated data points.

Above we have assumed implicitely that the lifetimes of erroneously measured sets remain WEIBULL distributed as found in practice. Even the fit of a WEIBULL distribution to such data points can be really good which is shown by investigations to be discussed in Sections IV.5. to IV.8..

Whilst testing the simulation program it appeared that the results with equal class sizes (n = 112 to 1,928) as calculated above do not differ with the results if n = 1,000 is taken in every case independently of the value of β and μ. Then we have per set of parameters:

1,000 data points with coordinates $\{T_{i.\Phi}, S(T_i)\}$.

Since the survival curves of this model are assumed to be straight lines in a ln t versus ln{-ln S} = ln H grid, the WEIBULL parameters (μ_{Φ}, β_{Φ}) can be estimated by means of a linear regression technique. The method is in line with our model and in agreement with cause and effect. We shall return to the causality based argument later in this Chapter where the robustness of parameter estimation is demonstrated for the case under consideration.

The degree of data dispersion due to measurement errors can be simulated by using differently valued coefficients of variation, v_s, of the error term. Once a given (chosen) coefficient of variation is applied to each T_i, the standard deviation changes proportionally to the value of T_i. For example, when the value of T_{650} = 15 years and v_s = 0.2, it follows that the related standard deviation σ_s amounts:

$\sigma_s(T=15)$ = 0.2 x 15 = 3 years and, consequently:

$\sigma_s(T=1)$ = 0.2 x 1 = 0.2 years.

Thus, the greater the time span, the greater the error dispersion and vice versa which is reasonably in agreement with observations in the real world. In order to express the degree of error, the following range of coefficients of variation is employed:

v_s = 0.10; 0.20 and 0.30

To get an idea of what this means, a value of T_i = 25 years is taken as an example. With v_s = 0.20 the standard deviation of the roughly normally (≈ CAUCHY) distributed $T_{i.\Phi}$ amounts to 25 x 0,20 = 5 years. This implies that the 95% probability area of $T_{i.\Phi}$ is approximately 15-35 years which relates to a high degree of data dispersion due to (quasi) measurement errors.

As the estimated β_Φ results from a regression with an error in the abscissa-variable, it can be expected that a higher degree of retirement data $\{T_{i.\Phi}, S(T_i)\}$ dispersion will result in a lower value of β_Φ; see, e.g., CRAMER (1969, page 139) and LANCASTER (1990). The question is how much the parameters will change in response to measurement errors in comparison with the parameters of the related hypothetical WEIBULL distribution. We have tried to answer that question by means of simulation.

IV.3.1. Simulation Results with defined Data Errors.

As above, the simulation is carried out with 3 different parameters:

- shape parameter of hypothetical distribution, β = 2; 4 and 6
- size parameter " " " , μ = 10; 25 and 50 years
- coefficient of variation applied to each T_i , v_s = 0.10; 0.20 and 0.30

This results in 27 data sets referring to:

μ_Φ, β_Φ and r (coefficient of determination with reference to regression technique). Furthermore, the ratios μ/μ_Φ and β/β_Φ are calculated.

Each simulation run with a given set of parameters was repeated twenty times in order to judge the consistency of the results.

Next, the data obtained from the 20 runs with a given set of parameters (v_s, μ and β) were averaged. These results are presented in Table IV-5 on the next page. Note that the value of r is high if v_s = 0.1 but decreases with increasing v_s and β. The decrease in r is less when the value of μ is higher. As shown later in Section IV.4.5., the coefficients of determination resulting from a regression technique applied to raw empirical retirement data are, generally, significantly higher. Probably, this is the effect of some data grouping by which measurement errors are smoothed out. For that phenomenon the reader is referred to the specialist literature, e.g., LANCASTER (1990).

v_s	r	μ	μ_Φ	μ/μ_Φ	β	β_Φ	β/β_Φ
0.100	0.992	10	9.873	1.01289	2	1.966	1.01743
0.200	0.968	10	9.488	1.05392	2	1.866	1.07188
0.300	0.923	10	8.791	1.13754	2	1.697	1.17827
0.100	0.979	10	9.939	1.00618	4	3.839	1.04206
0.200	0.917	10	9.746	1.02604	4	3.381	1.18309
0.300	0.819	10	9.373	1.06689	4	2.704	1.47931
0.100	0.958	10	9.992	1.00085	6	5.500	1.09087
0.200	0.849	10	9.927	1.00735	6	4.363	1.37518
0.300	0.712	10	9.873	1.01291	6	3.026	1.98276
0.100	0.994	25	24.730	1.01093	2	1.975	1.01261
0.200	0.973	25	23.894	1.04630	2	1.896	1.05488
0.300	0.935	25	22.436	1.11430	2	1.747	1.14492
0.100	0.985	25	24.940	1.00240	4	3.884	1.02981
0.200	0.941	25	24.686	1.01274	4	3.558	1.12428
0.300	0.864	25	24.229	1.03184	4	2.975	1.34455
0.100	0.972	25	25.080	0.99680	6	5.656	1.06074
0.200	0.893	25	25.294	0.98836	6	4.793	1.25185
0.300	0.777	25	25.696	0.97291	6	3.632	1.65215
0.100	0.994	50	49.518	1.00973	2	1.976	1.01218
0.200	0.976	50	48.055	1.04047	2	1.910	1.04724
0.300	0.939	50	45.215	1.10583	2	1.776	1.12589
0.100	0.988	50	49.892	1.00217	4	3.906	1.02406
0.200	0.952	50	49.595	1.00816	4	3.624	1.10365
0.300	0.887	50	49.163	1.01703	4	3.158	1.26650
0.100	0.978	50	50.083	0.99834	6	5.752	1.04320
0.200	0.914	50	50.723	0.98575	6	5.040	1.19047
0.300	0.813	50	52.126	0.95921	6	3.972	1.51075

Table IV-5: Results of erroneous data simulation.

The simulation results are evaluated graphically. Figure 7 on the next page represents the relationship between the shape parameter ratio β/β_Φ and the coefficient of variation with respect to simulated inaccuracy for three differently valued size parameters (μ = 10; 25 and 50 years). Figure 8 represents the size parameter ratio μ/μ_Φ and the coefficient of variation with respect to simulated inaccuracy for the three differently valued size parameters.

Figure 7 shows clearly that the shape parameter ratio increases with the degree of inaccuracy and with the value of the shape parameter of the hypothetical WEIBULL distribution. The shape parameter ratio decreases when the value of the size parameter of the hypothetical WEIBULL distribution increases. The ratio $\beta/\beta_\Phi \approx 2$ for $\beta = 6$ and $\mu = 10$ years is an extreme inaccuracy resulting in a very poor fit ($r = 0.712$).

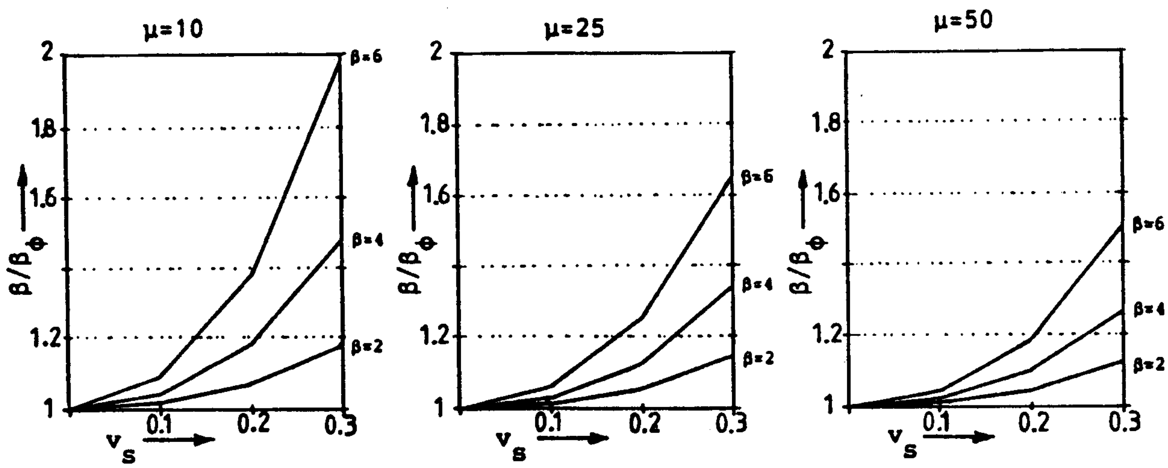


Fig. 7: Relationship between the shape parameter ratio and the coefficient of variation with respect to inaccuracy.

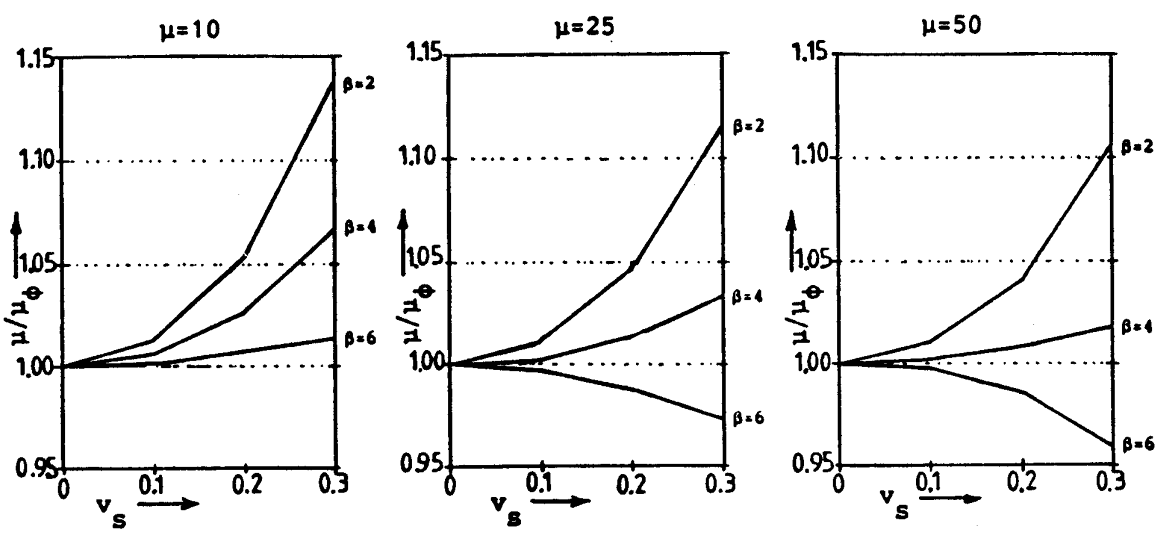


Fig. 8: Relationship between the size parameter ratio and the coefficient of variation with respect to inaccuracy.

Figure 8 shows that the size parameter ratio is larger than one for μ = 10, 25 and 50 years for $\beta \leq 4$. For μ = 25 and 50 years, the size parameter ratio becomes slightly less than one if $\beta = 6$. The plus or minus deviation from a size parameter ratio of one increases with the degree of inaccurancy in every case, but that deviation is relatively small. Generally, the size parameter ratio is much less influenced by measurement errors in empirical retirement data than the shape parameter ratio. The underestimate of the size parameter is relatively insignificant if $\beta > 2$ as is the overestimate if, say, $\beta < 6$ and $\mu > 20$ years.

The underestimate of the shape parameter due to inaccuracy is significant. This implies that the value of a (the duration of Phase II) is also underestimated if calculated according to our model:

$$a = \mu_3^{\frac{\beta_3-2}{\beta_3-1}} \tag{III/8}$$

For example, when μ_3 = 20 years and $\beta_3 = 4$, it follows that a = 7.37 years. If the shape parameter decreases due to measurement errors, to $\beta_3 = 3$, the value of a decreases to 4.47 years which is roughly 40% less. Accordingly, the probability of survival at t = a is overestimated, because it applies that:

$$S(a) = \exp[-(a/\mu_3)^{\beta_3}] = \exp[-\mu_3^{-\beta_3/(\beta_3-1)}] \tag{3}$$

For example, S(a) = 0.981748 for $\beta_3 = 4$ and μ_3 = 20 years and also for $\beta_3 = 3$ and μ_3 = 14.337423 years. S(a) increases to 0.988882 for $\beta_3 = 3$ and μ_3 = 20 years. The increase is not great, although the value of a changed by roughly 40% in this case.

Another kind of error is not a relative but an absolute one, e.g., an absolute error in the recorded lifetimes. That problem can easily be solved by introducing a location parameter; this will be discussed and applied in Section IV.7.1..

The quantitative insights achieved by this simulation are useful for interpreting the parameter estimates based on the raw empirical retirement data.

IV.4. Testing Procedure

The aim is to test the form of the distribution(s) in our probabilistic lifetime model rather than the parameters. The latter is a necessary by-product of testing. We have chosen for parametric testing which involves graphical and analytical techniques. The linear plot of our model as illustrated by Figure 6 and elaborated in Chapter III is an obvious tool that corresponds with preliminary graphical analyses of raw empirical retirement data.

The first step is to order the observed empirical lifetimes and to estimate the associated probability of survival by means of two methods. The ordinary one starts with a frequency histogram (the number of discards measured at a certain point in time divided by the total number of discards observed during a given time interval). This is a preliminary and informal judgement of empirical data properties in the light of our model. In this, censoring and other statistical problems are disregarded because they are not sufficiently known. Besides, some of the available empirical retirement data sets are given in failure percentages or fractions of the original units retired during each age interval. Subsequently, the survivor functions, $\hat{S}(T_j)$, are determined. These empirical survivor functions are then no longer independent estimates. Therefore empirical survivor curves have a tendency to smooth out the inherent fluctuations in the data but this is not a serious problem for a preliminary graphical analysis of raw empirical retirement data. Useful information can be gained by plotting $\ln[-\ln S(t)] = \ln H(t)$ against $\ln t$. Misspecification of a WEIBULL distribution will show curvature in the corresponding plot. Possibly, an n-component (composite) distribution with differently valued parameters would come into view if the relevant subpopulations were present.

The other method applied is that of KAPLAN & MEIER (1958). This uses the nonparametric product limit (PL) estimate of the survival function which can be interpreted as the following maximum likelihood estimator:

$$\hat{S}(t) = \prod_{j=1}^{k} \left(1 - (d_j/n_j)\right) \tag{4}$$

where:

$\hat{S}(t)$ = sample estimate of the survival function with lifetime variable t

n_j = number of capital units at risk at T_j, that is, the number of capital units alive and uncensored just prior to T_j

d_j = number of discards at T_j; more than one discard at T_j is allowed

n = number of capital units under observation with $k \leq n$ distinct lifetimes $T_1 < T_2 < \ldots . < T_k$ at which discarding occurs

If there is no censoring, it follows that:

$$n_1 = n \quad , \text{ and } \quad n_j = n_{j-1} - d_{j-1} \qquad (j = 2, \ldots ., k)$$

This reduces the KAPLAN-MEIER estimate (PL) to the ordinary empirical survival function.

$\hat{S}(t)$ is a step function that equals 1 at $t = 0$ and drops by a factor, $(n_j - d_j)/n_j$, immediately after each lifetime T_j. The estimate does not change at censoring times, however, the effect is felt in the values of n_j and hence in the sizes of the steps in $\hat{S}(t)$.

Since the integrated hazard is defined as $H(t) = -\ln S(t)$, the KAPLAN-MEIER estimate can also be written as:

$$\hat{H}(t) = -\ln \hat{S}(t) = -\sum_{j=1}^{k} \ln \left(1 - (d_j/n_j)\right) \tag{5}$$

After a log-conversion the values obtained by formula (5) can be used directly for plotting $\ln \hat{H}(T_j) = \ln[-\ln \hat{S}(T_j)]$ against $\ln T_j$.

The KAPLAN-MEIER method requires information on the number of discards at a given point in time and the number at risk. This information is not always available and, occasionally, it cannot be made available. For instance, in the case of roads, sewage systems and underground cables, etc., there is no question of physical units. For those cases where the number of observed units is unknown or the idea of units is nondescriptive, it was decided to employ 1,000 units as an imaginable but workable number. That particular number was chosen on the basis of computer simulation runs with 100, 1,000 and 10,000 units. This approach offers the possibility of recording discards in terms of initial investment (in constant prices), i.e., in "units of capital" instead of "physical units". Among the sets of empirical retirement data we have encountered such records as indicated in Table IV-3. (Appendix VII.1., page 2).

The following subsections are devoted to the testing procedures and to the estimation of the relevant lifetime distribution parameters.

IV.4.1. Preliminary Graphical Analysis

Empirical retirement data analysis is frequently undertaken by using graphical techniques, in particular, when mixed or composite distributions are expected. KAO (1959) and numerous other authors suggest a graphical method for a mixed WEIBULL model consisting of two subpopulations. NELSON (1969) stated that a plot of data provides a complete and easy-to-grasp picture, a convenient means of fitting a distribution to data, and allows one to assess whether a chosen theoretical distribution adequately fits to the data. A plot is also useful for looking at the data properties in general. NELSON introduced a method for hazard plotting of incomplete failure data, and also made use of the nonparametric KAPLAN-MEIER method for obtaining the product limit estimates. A graphical estimation of mixed WEIBULL parameters for ungrouped multicensored data is developed by NATESAN & JARDINE (1986). Although our WEIBULL model is not a mixed but a 3-component (composite) model, the graphical method for separating the subpopulations, if any, and finding the distribution parameters is in principle the same.

For the computation of the KAPLAN-MEIER estimates, the graphical presentation of the data points and the fitting of the (distinctive) survivor curves, we have made use of a flexible and powerful interactive system called MATHCAD.

The first step was to plot the data points $\{\ln \hat{H}(T_j)\}$ against $\ln T_j$ in the corresponding WEIBULL grid to obtain a graphical presentation on the monitor. If our hypothesis on the basis of an n-component (composite) WEIBULL concept is correct, the plotted data points will follow "n" straight lines. Each line (segment) reflects a subpopulation of our lifetime model. Before the "n" straight lines are constructed, the subpopulations are segregated, if present, actually by eye. Figure 9 on the next page shows the plotted data points for empirical retirement data of passenger cars in the Netherlands (Code P.C.NL) as an example. The data are individual observations (classwidth: one year).

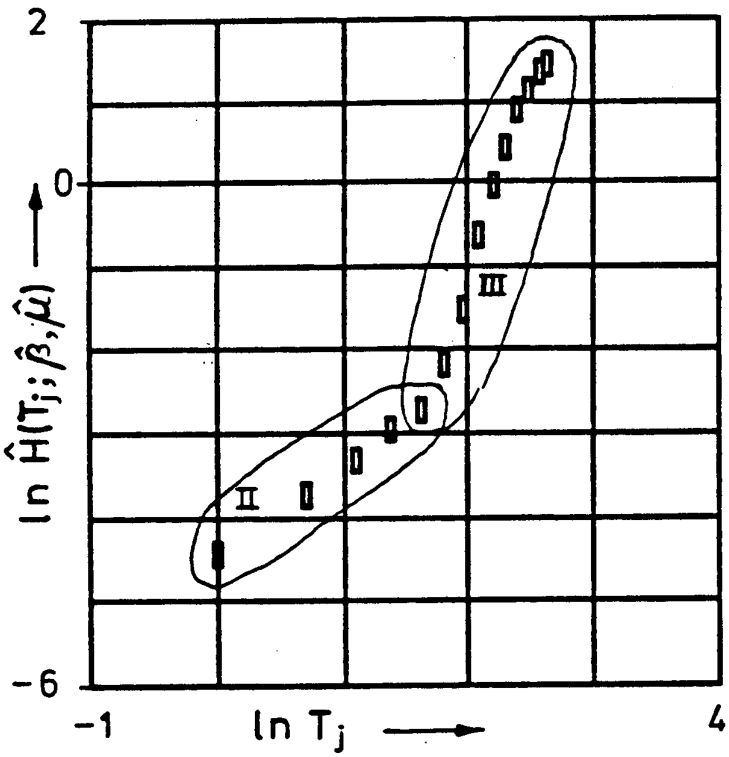


Fig.9: Plot of empirical retirement data with $y = \ln \hat{H}(T_j)$ and $x = \ln T_j$.

Obviously, this plot of empirical retirement data reflects subpopulation II (data points 1 to 5) and subpopulation III (data points 5 to 14). Both subpopulations have point 5 (encircled) in common which is the point nearest to the intercept of two straight lines (survivor curves) with differently valued parameters. As a result of this graphical representation of data points we have two sets which can be analyzed separately. The picture is very informative in the sense that "outliers" or any peculiar patterns, groups or curvatures come into view and can be treated appropriately. This is what a preliminary analysis by eye means; it has a diagnostic function in further testing and in estimating relevant parameters.

IV.4.2. Quasi-LS Parameter Estimation

As described in Section III.5. by function (III/21) and graphically presented in Section III.6., our model is based on WEIBULL integrated hazard functions. One way to test the acceptability of this model starts from the KAPLAN-MEIER estimates. Since empirical retirement data are raw, we have:

$$\ln \hat{H}(t) = \beta.\ln t - \beta.\ln \mu + u_H \tag{18}$$

where u_H is an error term which is usually normally distributed. Since (18) is a linear function, parameters β and μ can be estimated by means of a linear regression technique. Then we have:

$$e_H = \ln H(t) = \hat{\beta}.\ln t - \hat{\beta}.\ln \hat{\mu} \tag{19}$$

where:

e_H = error terms and residuals

$\hat{\beta}$ = shape parameter estimate

$\hat{\mu}$ = size parameter estimate.

For testing the acceptability of our model, we lay more emphasis on misspecification and on what we have termed S-discrepancies, defined as the difference between:

$\hat{S}(T_j) = \exp[-(T_j/\hat{\mu})^{\hat{\beta}}]$ and $\hat{S}(T_j) = \exp[-\hat{H}(T_j)]$

where $\hat{H}(T_j)$ is the KAPLAN-MEIER estimate which is directly obtained from the empirical retirement data.

Misspecification is tested on the basis of the integrated hazard functions as is discussed in Section IV.4.4.. For that test we have made use of two differently obtained sets of parameters, one set using the ML-method and the other set using the quasi-linear regression method discussed above. Since the empirical retirement data are converted to $\ln \hat{H}(T_j)$ and $\ln T_j$, we may call the latter a "quasi-least quares" method. According to NATESAN & JARDINE (1986) the interpretation of the value of the coefficient of determination resulting from this "quasi" linear regression technique differs from the one commonly used. They apply the following criteria for the goodness of fit which is, broadly speaking, a kind of KOLMOGOROV-SMIRNOV approach:

- Coefficient of determination $r > 0.95$, and
- Maximum discrepancy between any $\hat{S}(T_j) = \exp[-(T_j/\hat{\mu})^{\hat{\beta}}]$ and $\hat{S}(T_j) = \exp[-\hat{H}(T_j)]$ (as calculated on the basis of the estimated parameters) must be less than 20%. This is equivalent to any S-discrepancy of less than 0.2 (positive or negative).

We employ these criteria to judge the goodness of fit as far as curve fitting and parameter estimation on the basis of the quasi-linear regression technique are concerned.

Returning to the example presented above in Section IV.4.1. and applying the quasi-linear regression technique, we obtain two straight lines representing two subpopulations. This is demonstrated in Figure 10 below.

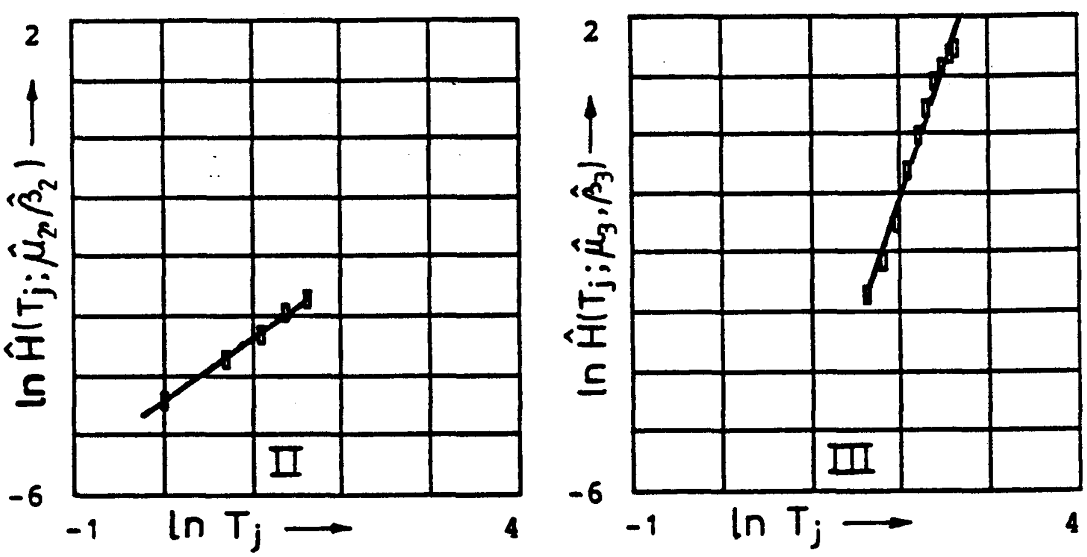


Fig. 10: Plots of empirical retirement data and the corresponding survivor curves for subpopulation II (left) and III (right).

From the curve fitting technique for each subpopulation, the following data are estimated:

- WEIBULL shape parameter $\hat{\beta}$ which is the slope of the corresponding straight line;
- WEIBULL size parameter $\hat{\mu}$ which follows from $\hat{\mu} = \exp(T_\mu)$, where T_μ is the point in time for which $\ln H = b + \beta \ln T = 0$; in other words: $\hat{\mu} = -b/\hat{\beta}$, where b is the intercept of the regression line.
- coefficient of determination r of the linear regression.

Since the core WEIBULL distribution (Phase III) is most important, it is checked whether or not the visually selected group of data points related to Phase III is correct. For that purpose the coefficient of determination is maximized by including more or less data points which are related to Phase II. This is an easy iterative procedure which leads quickly to the estimates of the shape and size parameters of the core WEIBULL distribution. Of course, this kind of parameter estimation has its objections. At this stage, however, parameter estimation is less important than testing of the model fit and separating of data points into their corresponding subpopulations and phases. We have applied the preliminary graphical analysis discussed above as well as the decomposition of subpopulations, if any, and the preliminary parameter estimation according to the ordinary quasi-linear regression method in all cases, i.e., for all sets of capital assets listed in Tables IV-1, IV-2, IV-3 and IV-4. The informal results are summarized in Appendix VII.3., page 1 and 2. From these 97 sets there are 30 representative sets selected for a more detailed analysis. For that purpose the ML-method is additionally employed to estimate the parameters of the graphically selected groups of data points related to the core lifetime distribution. A calculation of the S-discrepancies for both estimating methods (quasi-LS and ML) will answer the question which one is preferred. Where possible and meaningful, a Chi-square test is additionally employed (see page 126).

IV.4.3. ML Parameter Estimation

Many authors have considered methods based on the likelihood function for complete as well as for censored data. COX and OAKES (1984) state that Type II censoring is an efficient technique in industrial life-testing. In Type II censoring, observation ceases after a predetermined number d of discards. A crucial condition is that, conditionally on the values of any explanatory variables, the lifetime forecast for any unit which has survived to $c_i < T_i$ should not be affected if the unit in question is censored at time c_i. According to COX and OAKES, the above condition is satisfied if the potential censoring times are random variables c_i which are independent of the T_i. Since Type II censoring is more general and can depend on the past history of the entire discarding process, we have adopted this scheme. Below we may follow the approach of COX and OAKES (1984), pages 32 to 38.

The likelihood function from n independent units, indexed by i, has the following form:

$$Lik = \Pi_u f(T_i;\phi).\Pi_c S(c_i;\phi) \qquad (6)$$

where:

$f(T_i;\phi)$ = density of discarding at T_i for i = 1, 2, ...n

$S(c_i;\phi)$ = probability of survival beyond c_i

c_i = random variable of potential censoring times

Function (6) consists of two products which are taken over uncensored (index u) and censored (index c) units. The characteristics of the distribution are in this likelihood function expressed by vector parameter ϕ. In terms of observed discarding or censoring time, $x_i = \min(T_i, c_i)$, the log likelihood becomes:

$$\ln (Lik) = \Sigma_u \ln f(x_i;\phi) + \Sigma_c \ln S(x_i;\phi) \qquad (7)$$

Since $f(x_i;\phi) = h(x_i;\phi).S(x_i;\phi)$, and thus:

$\ln f(x_i;\phi) = \ln h(x_i;\phi) + \ln S(x_i;\phi)$, function (7) may be written:

$$\ln (Lik) = \Sigma_u \ln h(x_i;\phi) + \Sigma_u \ln S(x_i;\phi) + \Sigma_c \ln S(x_i;\phi) \qquad (7a)$$

When the second and the third term are combined, function (7a) becomes:

$$\ln (Lik) = \Sigma_u \ln h(x_i;\phi) + \Sigma \ln S(x_i;\phi) \qquad (8)$$

Since the second term of (8) is the "minus integrated hazard", the log likelihood function becomes:

$$\ln (Lik) = \Sigma_u \ln h(x_i;\phi) - \Sigma H(x_i;\phi) \qquad (9)$$

The fundamental role played by the hazard function in the log likelihood function is evident.

In the case under consideration the distribution is assumed to be a WEIBULL model with unknown parameters. Then the vector parameter ϕ consists of shape parameter β and size parameter μ. Differentiating the log likelihood function with respect to β and μ in turn and equating to zero, we obtain the estimating equations:

$$U_\beta = \partial(\ln Lik)/\partial\beta = 0, \text{ and: } U_\mu = \partial(\ln Lik)/\partial\mu = 0 \qquad (10)$$

For an EXPONENTIAL distribution (WEIBULL with parameters $\beta = 1$ and μ_2), the maximum likelihood estimator of $\hat{\mu}_2$ becomes:

$$\hat{\mu}_2 = (1/d)\Sigma T_i \qquad (11)$$

where d is the total number of discards and the summation term is the total of the censored <u>and</u> uncensored failure times.

The log likelihood function for a WEIBULL distribution can be obtained from (9) by substituting:

$h(x_i;\phi) = (\beta/\mu^\beta).T_i^{\beta-1} = h_w(T_i)$,and $H(x_i;\phi) = (T_i/\mu)^\beta = H_w(T_i)$

Then we obtain for a WEIBULL distribution for d discards:

$$\ln (Lik) = d.\ln \beta - \beta.d.\ \ln(\mu) + (\beta - 1)\Sigma_u \ln T_i - (1/\mu)^\beta \Sigma T_i^\beta \qquad (12)$$

Differentiating (12) with respect to μ and equating to zero, gives:

$$U_{\mu} = -d.\beta(1/\mu) + \beta(1/\mu)^{\beta+1}\Sigma\, T_i^{\beta} = 0 \tag{13}$$

The maximum likelihood estimator $\hat{\mu}$ can be found explicitly from (13) when $\hat{\beta}$ is specified:

$$\hat{\mu} = [(1/d)\Sigma\, T_i^{\hat{\beta}})^{1/\hat{\beta}} \tag{14}$$

This result could be derived directly from (11) because (T^{β}) has an EXPONENTIAL distribution with parameter (μ^{β}).
Differentiating (12) with respect to β and equating to zero, gives:

$$U_{\beta} = (d/\beta) - d.\ln\mu + \Sigma_u \ln T_i - (1/\mu)^{\beta}\,\Sigma\, T_i^{\beta}.\ln\,(T_i/\mu) = 0$$

Substitution of (14) into the above yields the following equation:

$$U_{\beta} = (d/\hat{\beta}) + \Sigma_u \ln T_i - (d/\Sigma\, T_i^{\hat{\beta}})\Sigma\, T_i^{\hat{\beta}}.\ln T_i = 0 \tag{15}$$

Equation (15) can be solved by a one-dimensional iterative scheme in $\hat{\beta}$.
If the empirical retirement data are uncensored, estimating equations (14) and (15) use only the total of the observed uncensored failure times. For n complete data we obtain:

$$U_{\beta} = (n/\hat{\beta}) + \sum_{i=1}^{n} \ln T_i - (n/\sum_{i=1}^{n} T_i^{\hat{\beta}})\Sigma\, T_i^{\hat{\beta}}.\ln T_i = 0 \tag{16}$$

$$\text{and: } \hat{\mu} = [(1/n)\sum_{i=1}^{n} T_i^{\hat{\beta}})^{1/\hat{\beta}} \tag{17}$$

With estimating equations (16) and (17) the ML-estimators can be determined. This is done with 30 representative sets which are selected from preliminary analyses. The results will be discussed in the course of this chapter.

IV.4.4. Testing of Misspecification by Plotting Techniques.

Evidence may be given that the WEIBULL concept is not misspecified by the neglect of random multiplicative heterogeneity in the hazard function. Then the survival function is:

$$S(T) = \int_0^{\infty} \exp[-(T/\mu)^{\beta}.v]dG(v)$$

where $\ln[-\ln S(t)]$ is not a straight line but a concave curve.
Many authors have dealt with generalized errors and residuals as defined by COX & SNELL (1968). If a plot of log integrated hazard against log

failure time produces a straight line (survivor curve), a WEIBULL distribution is called for. Then as a second stage, the approximate form of dependence on the explanatory variables can be examined, leading, if formal model fitting is desirable, to such a parametric model.
Since various implicit and explicit assumptions in the analysis of individuals are made, it is important to examine misspecification of the assumed model. If our model is an adequate representation of a WEIBULL distribution of (ordered) lifetimes, T_j, and if T_j has survival function $S(t|\beta,\mu)$, then $\hat{S}(T_j)$ is uniformly distributed and $-\ln \hat{S}(T_j) = \hat{H}(T_j)$ has a unit EXPONENTIAL distribution. As an overall check on the model,

$$S_o(t) = \exp[-H_o(t)] \tag{18}$$

the ordered values of $\hat{H}(T_j)$ obtained by the KAPLAN-MEIER method may be plotted against their expected values for the unit EXPONENTIAL distribution. As an example, Figure 11 below is a presentation of the plot meant here.

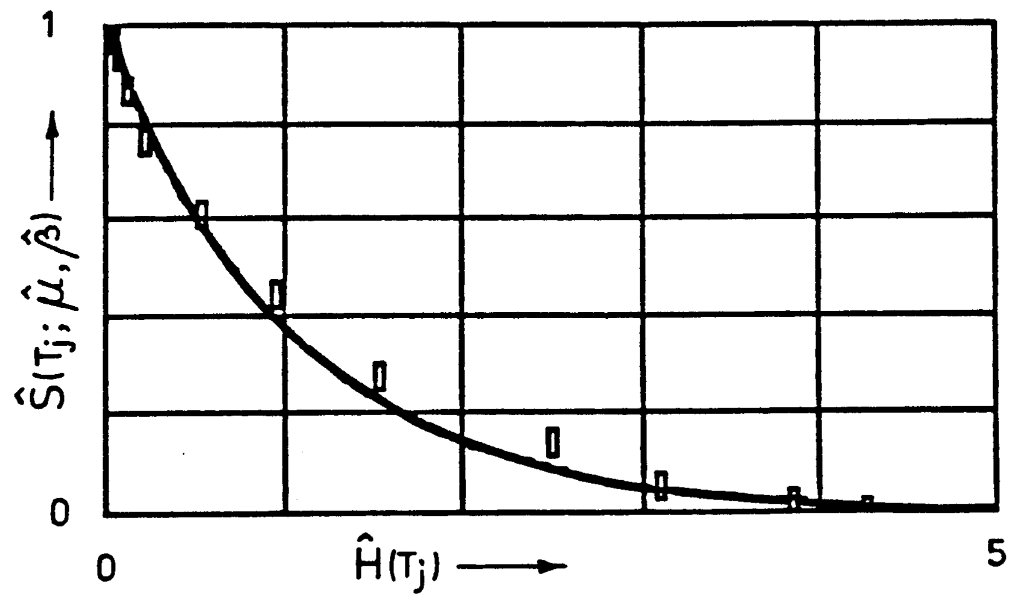


Fig. 11: Plot of empirical retirement data points of individuals and the unit EXPONENTIAL distribution curve.

The behaviour in the tails of this unit EXPONENTIAL distribution is not clearly represented in the plot above. In order to get more insight into the tail on the right side, the y-axis is taken as -ln S instead of S. Then the EXPONENTIAL curve becomes a straight line starting from the origin with a slope of 45°. This transformation is illustrated in

Figure 12 below; one plot on the left for all data points and one on the right for those data points for which apply that $\hat{H}(t;\hat{\beta},\hat{\mu}) \leq 1$.

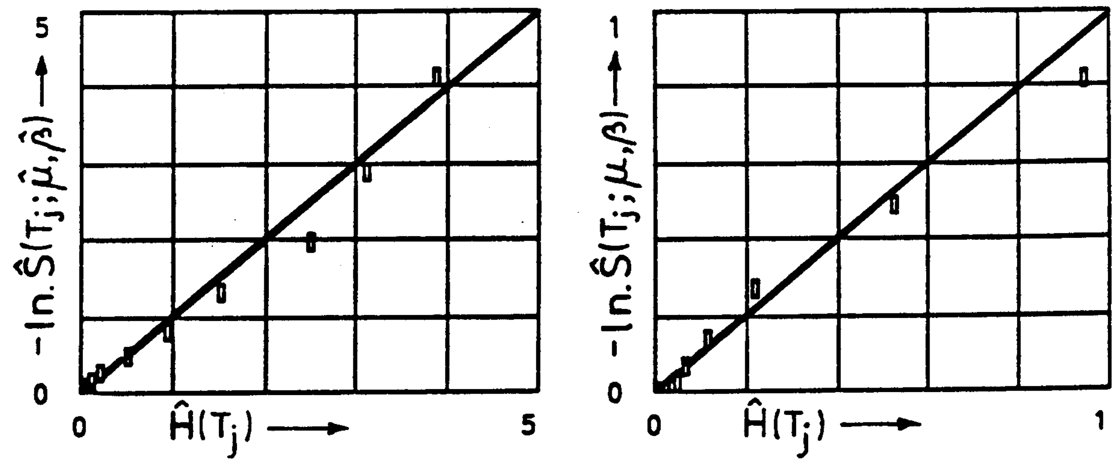


Fig. 12: Plot of empirical retirement data points $\{-\ln \hat{S}(T_j;\hat{\beta},\hat{\mu})$ against $\hat{H}(T_j)\}$. Left: all data points. Right: data points with H ≤ = 1.

In the left plot outliers related to the right tail of the distribution are magnified whereas outliers in the centre of the distribution come into view in the plot on the right hand side. The area $0 < H(t) < 1$ is the most interesting part of the distribution as will become clear in Chapter V..

The plots discussed above give no clear information about the left part of any lifetime distribution model, that related to Phases I and II, because the variance of the plotted points increases as $t \to 0$. AITKIN & CLAYTON (1980) solved that problem of fitting Exponential, Weibull and Extreme Value distributions to complex (censored) survival data by plotting the following variance-stabilized transformation:

$$\sin^{-1}[S(t)]^{\frac{1}{2}} \quad \text{against} \quad \sin^{-1}.\exp[-\{H(t)/2\}]$$

In a plot this is a straight line with a slope of 45° wherein the data points coordinates are:

$$\sin^{-1}[\hat{S}(T_j;\hat{\beta},\hat{\mu})]^{\frac{1}{2}} \text{ against } \sin^{-1}.\exp[-\tfrac{1}{2}\hat{H}(T_j)]$$

Figure 13 on the next page gives an example of a variance-stabilized plot of empirical retirement data points.

It should be noted that the data points in the variance-stabilized plot are reversed in order, i.e., the points in the upper right corner of Fig. 13 are related to the left part of Fig. 12, and the points near the origin of Fig. 13 are related to the right tail of the distribution.

In this plot outliers at both ends of the distribution are clearly seen. We have encircled these outliers, if any, in the further plotting of points for the 30 selected sets of empirical retirement data. We return to this subject later when the results are discussed. Generally, the variance-stabilized residual plotting appeared to be very useful in judging misspecification and to check whether data or data groups are correctly decomposed.

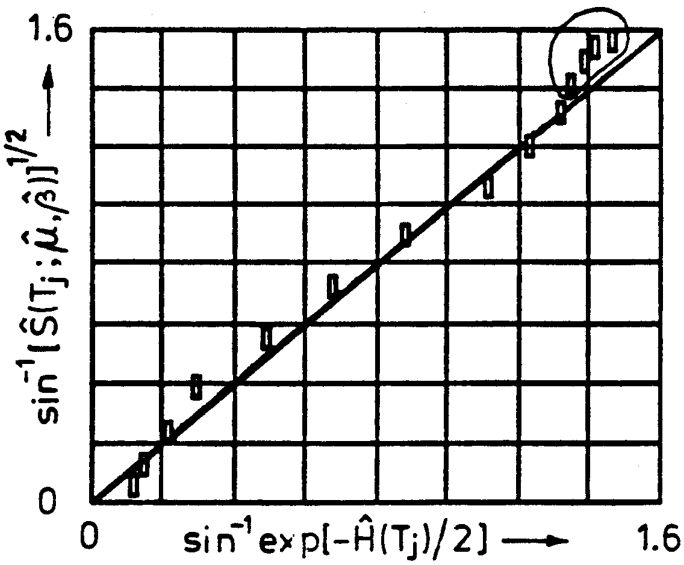


Fig.13: Variance-stabilized residual plotting of empirical retirement data points.

H-residual plotting is applied on the basis of the quasi-LS parameter estimation method and of the ML-method. In the case of the quasi-LS-method the empirical survival estimates are determined using the KAPLAN-MEIER method. Residual plots according to Figures 11 to 13 are made for the 30 selected and representative sets of empirical retirement data. Each set is printed on one page with on the left hand side the results of the KM-method (KAPLAN-MEIER), and on the right hand side the ML-method of parameter estimation. These pages are inserted in Appendices VII.4.1., VII.5.1., VII.6.1. and VII.7.1., and will be considered later.
For those empirical retirement sets in which subpopulation II can be seen, H-plots are made for Phase II only. According to our model it is expected that $\beta_2 = 1$ and therefore a straight line with a slope of 45° is fitted to the data points with coordinates:

$x = \hat{H}(T_j)$ from to the KAPLAN-MEIER estimates and for $j = 1, 2, \ldots k$ data points of Phase II whereas the k-th data point is also related to Phase III

$y = -\ln \hat{S}(T_j) = T_j/\hat{\mu}_2$, where $\hat{\mu}_2$ is the point in time for which $\ln \hat{H}(\hat{\mu}_2) = 0$.

The curve-fitting procedure is such that the sum of the S-discrepancies:

$$\sum_{j=1}^{k} \{\exp[-\hat{H}(T_j)] - \exp[-(T_j/\hat{\mu}_2)]\},$$ is minimized.

A H-plot, $-\ln \hat{S}(T_j;\hat{\mu}_2)$ against $\hat{H}(T_j)$, for Phase II related to the empirical retirement data set of Code M.2.1. is shown by Figure 14 below.

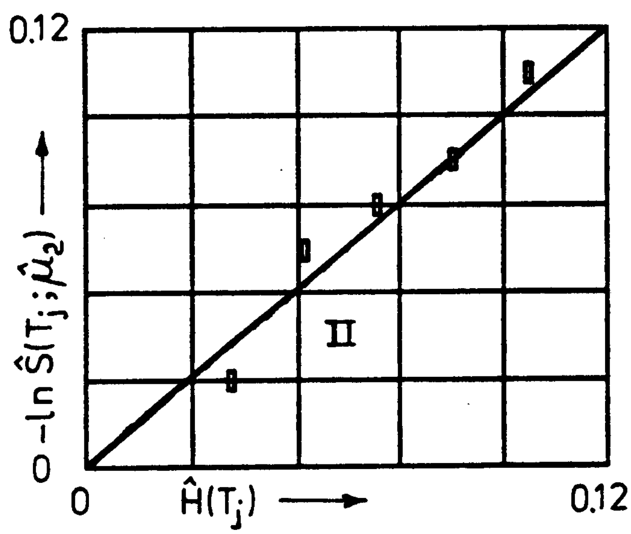


Fig. 14: H-plot, $-\ln \hat{S}(T_j;\hat{\mu}_2)$ against $\hat{H}(T_j)$, for Phase II related to Lathes (Code M.2.1.)

The H-plots referring to the remaining 12 sets of individual empirical retirement data for which Phase II can be seen are inserted in Appendix VII.2.1., pages 1 to 4. Note that the H-scale applied in these plots is such that the deviation in the y-direction between the data points and the straight line with a slope of 45° is numerically very small. All points are associated with a moderately to very high value of the probability of survival in time interval (0,a). This is one of the reasons that the parameters for the left tail of the distribution cannot be efficiently estimated by analytical change points methods. See e.g. PRAAGMAN (1986) and the references therein.

IV.4.5. Parameter Estimates

In Section IV.4.2. above we have defined S-discrepancies as follows:

$$\mathrm{Sdis} = \exp[-(T_j/\hat{\mu})^{\hat{\beta}}] - \exp[-\hat{H}(T_j)] \qquad (20)$$

We have chosen S-discrepancies because our lifetime model to be tested is based on the probability of survival of capital assets. The S-discrepancies are determined for all data points related to each of the 30 selected and representative sets.

Curve fitting (straight line with slope of 45°) to the empirical retirement data related to Phase II took place by reducing the sum of S-discrepancies of k data points to a minimum. This implies that the values of the S-discrepancies related to Phase II are known as well as the shape parameter, $\beta_2 = 1$, and the size parameter estimate $\hat{\mu}_2$. The duration of Phase II can be estimated on the basis of the probability of survival at $t = \hat{a}$:

$$S(\hat{a}) = \exp[-\hat{a}/\hat{\mu}_2] = \exp[-(\hat{a}/\hat{\mu}_3)^{\hat{\beta}_3}]$$

From the above it follows that:

$$\hat{a} = (\hat{\mu}_3^{\hat{\beta}_3}/\hat{\mu}_2)^{1/(\hat{\beta}_3-1)} \qquad (21)$$

As argued in Section IV.3., $\hat{\beta}_3$ can/will be underestimated due to measurement errors whereas the size parameters, $\hat{\mu}_2$ and $\hat{\mu}_3$, are not significantly effected as far as the measurement errors are symmetrically (≈ CAUCHY) distributed. Hence, it can generally be expected that:

$\hat{a} > a$ according to our model; see (III/6)

$\hat{\mu}_2 < \mu_2 = \mu_3^2$ according to our model; see (III/5)

As mentioned in Section IV.4.4. above, Phase II is apparent in 13 of the 30 selected representative sets of capital assets.

For Phase III the values of $\hat{S}(T_j;\hat{\beta}_3,\hat{\mu}_3)$ are determined by two differently obtained sets of parameter, one by the KM-method and quasi-linear regression, and the other by the ML-method. Every (what we have called) "S-discrepancy" of each set of data points was recorded and examined to determine the maximum discrepancy of outliers. Furthermore, the total and mean of each set of S-discrepancies were taken. The results for both methods of parameter estimation (KM and ML) are summarized in Table IV-10 on the second next page.

The S-discrepancy analysis deals with both goodness of fit and the most appropriate method of parameter estimation, KM or ML. The "most useful" method is defined as that which gives the best goodness of fit to the empirical retirement data. The estimated WEIBULL parameters according to our model elaborated in Chapter III are summarized in Table IV-11 on the second next page.

It is noted that the value of the parameters according to the KM-method as summarized in Table IV-11 are not or not always equal to the preliminary determined informal values as recorded in Tables IV-6 to IV-9 (Appendix VII.3., pages 1 and 2). The differences in value are minor; they are mainly due to refinements in the analysis. Furthermore, it is noted that the partition parameter between Phases II and III is denoted as, (a), and indirectly estimated from $\hat{\beta}_3$ and $\hat{\mu}_3$ by using (III/8). As stated above, (a) can/will be underestimated. It appeared that 17 sets listed in Table IV-11 contain no retirement data showing Phase II. Phase II is apparent for 13 sets of empirical retirement data listed in Table IV-12 below.

CODE	TYPE OF CAPITAL ASSET	Number of points k	Indirect Estimates			Direct Estimates		
			$(\mu_3)=\hat{\mu}_3^2$	(a)	$\hat{\mu}_2$	$\hat{a}$	Σ Sdis	$S(\hat{a})$
M. 1.1.	Milling equipment	2	177.493	6.797	142.857	7.192	0.004	0.951
M. 2.1.	Lathes	5	183.882	8.032	83.513	9.412	0.002	0.894
M. 6.1.	Surface-treating equipment	2	278.092	3.803	62.937	8.301	-0.020	0.876
D. 2.1.	Lorries and trucks	4	104.181	3.661	62.240	4.596	-0.0007	0.929
D. 6.3.	Pumps and compressors	2	313.593	3.476	127.660	5.783	-0.008	0.956
D. 7.1.	Electric generators	3	280.873	2.501	148.000	2.800	-0.001	0.981
D.10.2.	Machining equipment	3	551.169	1.707	52.457	12.039	0.0006	0.795
3-1	Water work pumps	2	556.238	3.459	207.317	6.297	-0.003	0.970
30-4	Mazda B-lamps (60W)	3	40.788	3.265	39.095	3.315	0.0001	0.919
38-1	Coal flat train cars	9	446.584	9.014	512.345	8.675	0.007	0.983
53-1	Rodger ballast train cars	8	482.948	6.311	167.335	9.682	0.0009	0.944
64-2	Automobiles 1900-1922	3	51.780	3.123	32.622	3.797	0.002	0.890
P.C.NL	Passenger cars NL	5	88.657	4.859	80.238	5.004	0.002	0.940

Table IV-12: Indirectly and directly estimated parameters referring to Phase II as found for 13 sets of empirical retirement data.

The shape parameter is not estimated but taken as $\beta_2 = 1$. In all 13 cases parameters μ_3 and β_3 followed (by chance) from the KM-method. Alternatively the size parameter $\hat{\mu}_2$ and partition parameter $\hat{a}$ are directly estimated by using (21).

CODE	KIND OF CAPITAL ASSET	S-residuals; KM-method			S-residuals; ML-method			MOST USEFULLY METHOD
		mean	max.	sum	mean	max.	sum	
M. 1.1.	Milling equipment	-0.00048	-0.037	-0.006	0.015	0.057	0.183	KM(0.998)
M. 2.1.	Lathes	-0.008	-0.057	-0.077	0.027	0.065	0.273	KM(0.994)
M. 4.1.	Grinding equipment	-0.008	-0.076	-0.100	0.011	0.077	0.146	KM(0.985)
M. 6.1.	Surface-treating equipment	0.002	0.082	0.025	0.063	0.153	0.692	KM(0.992)
D. 1.1.	Passenger and delivery cars	-0.008	-0.074	-0.110	0.003	0.063	0.037	ML
D. 2.1.	Lorries and trucks	0.010	0.106	0.134	0.024	0.128	0.331	KM(0.992)
D. 5.2.	Wrapping equipment	0.001	0.058	0.025	0.009	0.068	0.195	KM(0.997)
D. 6.3.	Pumps and compressors	0.002	-0.076	0.040	0.021	0.092	0.404	KM(0.994)
D. 7.1.	Electric generators	0.00052	0.043	0.014	0.004	0.055	0.108	KM(0.998)
D. 9.1.	Measuring and controlling equipment	0.018	0.059	0.736	0.007	0.054	0.309	ML
D.10.2.	Machining equipment	-0.004	-0.068	-0.101	0.034	0.108	0.817	KM(0.991)
3-1	Water works pumps	0.011	0.082	0.214	0.030	0.108	0.599	KM(0.988)
4-1	Water works steam engines	0.013	0.083	0.166	0.038	0.127	0.497	KM(0.984)
9-1	Central office equipment(telephone)	0.00029	0.097	0.007	0.00042	0.092	0.010	KM(0.995)
11-2	Aerial cables (telephone)	-0.00080	-0.009	-0.015	-0.00046	-0.014	-0.009	ML
14-1	Underground cables (telephone)	-0.008	-0.055	-0.218	-0.005	-0.033	-0.145	ML
24-5	Wooden poles (telegraph)	-0.001	-0.044	-0.221	-0.00058	-0.015	-0.013	ML
30-4	Mazda B-lamps (60W electric)	-0.007	-0.046	-0.049	0.029	0.073	0.202	KM(0.995)
33-1	Steam locomotives (rail road)	0.020	0.095	0.809	-0.002	0.041	-0.088	ML
34-1	Passenger cars (rail road)	0.003	0.058	0.166	-0.005	-0.046	-0.232	ML
38-1	Coal flat cars (rail road)	-0.003	-0.037	-0.060	0.005	0.033	0.121	KM(0.999)
44-6	Crossties (rail road)	0.003	-0.056	0.028	0.00090	0.055	0.010	ML
53-1	Rodger ballast cars (rail road)	-0.009	-0.167	-0.199	0.031	-0.096	0.684	KM(0.974)
56-1	Corn cultivators (1-row)	0.009	0.071	0.151	0.011	0.097	0.169	KM(0.990)
59-1	Grain binders (5 to 8-foot)	0.012	0.069	0.239	0.010	-0.072	0.206	ML
64-2	Passenger automobiles (1922)	0.004	0.025	0.040	0.020	0.068	0.196	KM(0.999)
D.NL12	Dwellings NL (12 points)	-0.002	0.068	-0.030	-0.020	0.089	-0.243	KM(0.998)
D.NL48	Dwellings NL (48 points)	0.001	0.118	0.055	0.003	0.155	0.152	KM(0.992)
P.C.NL	Passenger cars NL	0.012	0.055	0.121	0.027	0.082	0.275	KM(0.993)
B.T.NL	Bus tyres (The Hague, NL)	0.009	0.038	0.054	-0.004	0.006	-0.021	ML

Table IV-10.: Analyses of residuals $\hat{S}(t)$ (estimated) minus $\hat{S}(T_j)$ (measured), and decision between KM and ML method of parameter estimation for a WEIBULL core distribution.

CODE	TYPE OF CAPITAL ASSET	KM-estimating Method				ML-estimating Method		
		$\hat{\beta}_3$	$\hat{\mu}_3$	(a)	r	$\hat{\beta}_3$	$\hat{\mu}_3$	(a)
M. 1.1.	Milling equipment	4.848	13.323	6.797	0.998	5.08	13.50	7.13
M. 2.1.	Lathes	5.979	13.560	8.032	0.994	7.00	13.86	8.94
M. 4.1.	Grinding equipment	3.293	13.934	4.417	0.985	3.39	14.23	4.68
M. 6.1.	Surface-treating equipment	2.904	16.676	3.803	0.992	2.95	18.10	4.09
D. 1.1.	Passenger and delivery cars	2.386	6.729	1.701	0.995	2.31	6.91	1.58
D. 2.1.	Lorries and trucks	3.266	10.207	3.661	0.992	3.22	10.44	3.64
D. 5.2.	Wrapping equipment	1.790	13.159	0.503	0.997	1.78	13.41	0.48
D. 6.3.	Pumps and compressors	2.765	17.709	3.476	0.994	2.94	18.17	4.08
D. 7.1.	Electric generators	2.474	16.756	2.475	0.998	2.56	16.84	2.76
D. 9.1.	Measuring and controlling equipment	1.904	17.160	0.741	0.991	1.68	16.63	0.26
D.10.2.	Machining equipment	2.204	23.477	1.707	0.991	2.84	24.82	4.34
3-1	Water work pumps	2.646	23.585	3.459	0.988	2.52	24.50	2.97
4-1	Steam engines	3.847	33.717	9.798	0.984	3.71	34.84	9.40
9-1	Central office equipment	1.659	9.767	0.308	0.995	1.62	9.76	0.25
11-2	Aerial cables	2.575	10.902	2.392	1.000	2.63	10.90	2.52
14-1	Underground cables	2.379	15.955	2.140	0.996	2.75	15.93	3.27
24-5	Wooden poles	2.945	11.720	3.307	0.998	3.13	11.92	3.73
30-4	Mazda B-lamps (60W)	3.763	6.387	3.265	0.995	4.60	6.61	3.91
33-1	Steam locomotives	3.996	28.797	9.382	0.991	3.66	27.93	7.98
34-1	Passenger train cars	4.005	36.779	11.084	0.991	4.31	36.17	12.25
38-1	Coal flat train cars	4.581	21.133	9.014	0.999	4.85	21.27	9.62
44-6	Railway cross ties	6.990	11.749	7.787	0.988	6.21	11.80	7.35
53-1	Rodger ballast train cars	3.477	21.976	6.311	0.974	5.06	22.38	10.40
56-1	Corn cultivators (1-row)	3.218	14.425	4.330	0.990	2.91	14.60	3.58
59-1	Grain binders (5 to 8-foot)	2.557	16.343	2.718	0.988	2.42	16.38	2.29
64-2	Automobiles 1900-1922	3.365	7.196	3.123	0.999	3.74	7.37	3.55
D.NL12	Dwellings NL (12 points)	3.881	102.266	20.516	0.998	5.00	97.266	30.97
D.NL48	Dwellings NL (48 points)	3.547	93.469	15.733	0.992	2.98	95.304	9.52
P.C.NL	Passenger cars NL	4.389	9.416	4.859	0.993	4.60	9.567	5.11
B.T.NL	Bus tyres (The Hague, NL)	3.281	144,208	16,304	0.998	3.03	140,092	13,32

Table IV-11: Results of KM and ML parameter estimating methods with respect to the WEIBULL core distribution of lifetimes and to the duration of the pre-aging period (a).

When the values of the indirectly estimated parameters are compared with the directly estimated parameters, the overestimation of the size parameter and the underestimation of the shape parameter of the core distribution are obvious in 12 cases. With regard to Code 38-1, $\hat{a}$ is 3.8% less than (a) and $\hat{\mu}_2$ is 14.7% more than (μ_2).

According to Table IV-12, the value Σ Sdis is low in all cases. The maximum Sdis of any data point related to Phase II is not recorded; only one data point related to Code M.6.1. showed a maximum discrepancy of Sdis $\approx$ 0.02 which is low indeed. The goodness of fit is also demonstrated by the H-plots in Appendix VII.2.1., pages 1 to 4 and by Figure 12. More details are discussed in Sections IV.5. to IV.8..

IV.4.5.1. Discussion of Parameter Estimates

Generally speaking, the difference between KM and ML applied in parameter estimating with respect to the core distribution is negligible. The overall results obtained by the KM-method are slightly better. Bearing in mind the "goodness of fit" criteria as set by NATESAN & JARDINE (1986),

$r > 0.95$, and any S-discrepancy,

$\{\exp[-(T_j/\hat{\mu}_3)^{\hat{\beta}_3}] - \exp[-\hat{H}(T_j)]\} < 0.2$ (positive or negative)

as discussed in Section IV.4.2., the KM-method meets these criteria in all cases. According to Table IV-11, the lowest coefficient of determination r is 0.974; for 23 out of 30 sets the coefficient of determination is more than 0.99 and for one set (Code 11-2) it is even one. If no information matrix is needed, there is no need for the ML-method. For some cases considered here the parameter estimates according to the ML-method are equally good and for 10 out of 30 sets the ML-method gives a slightly better result than the KM-method. In all cases the mean of the Sdis is close to zero and also the sums are very low which is an indication for a good fit. It is noted that 12 out of 30 sets (40%) give a negative mean of the Sdis's when the KM-method is applied whereas 7 out of 30 sets (23.3%) when the ML-method is applied. This means that the ML-estimators result on average in a somewhat higher probability of survival than on the basis of the KM-estimates. The 10 sets for which the ML-estimators are slightly better consist of 6 sets with a negative and 4 sets with a positive sum. It is striking that there is one set (Code 11-2) with r = 1.00.

In the first instance it may be surprising that the KM-method of parameter estimation performed to be more efficient than the ML-method. However, several authors have demonstrated that for the most commonly values of the shape parameter of a two parameter WEIBULL distribution the ML-method is not so efficient. NEWBY (1980) showed analytically that the β and μ estimators based on the coefficient of variation and complete samples are, in general, highly asymptotically efficient and perform well compared with other estimators.

The modelled plots as in Figure 6 in Chapter III of the 30 selected and representative sets of empirical retirement data are shown in Appendices VII.4.2., VII.5.2., VII.6.2. and VII.7.2.. Generally, the modelled plots performed well and suggest that the WEIBULL distribution model fits the data points. Misspecification due to neglecting random multiplicative heterogeneity in the hazard as discussed by LANCASTER (1985/1990) and CHESHER & IRISH (1987) are practically inconceivable since the subpopulations with differently valued hazard parameters are segregated by means of preliminary graphical analyses. The plots of H-residuals (Appendices VII.4.1., VII.5.1., VII.6.1. and VII.7.1.) show that the WEIBULL core distribution is evident. The same holds for the EXPONENTIAL distribution found for Phase II and demonstrated by the plots of H-residuals in Appendix VII.2.1., pages 1 to 4.

More details will be discussed in the next Sections IV.5. to IV.8..

For informal testing results obtained by the application of the Chi-square method, we refer to the work of HINSKENS & VAN WIERINGEN (1989) conducted by BEKKER and DAAMS. A final Chi-quare test is inserted in Section IV.9. (Table IV-14).

IV.5. Results for Mechanically Operated Tools

The 8 sets of mechanically operated tools classified by different functions and discarded from one and the same engineering works are described above in Section IV.2. and listed in Table IV-1 (Appendix VII.1., page 1).
The results of a preliminary graphical analysis followed by segregation of subpopulations and informal parameter estimation of the WEIBULL core distribution are summarized in Table IV-6 (Appendix VII.3., page 1). A detailed description of these analyses and the plots of empirical data points are shown in the M.Sc.-thesis work of HINSKENS & VAN WIERINGEN (1989) directed by BEKKER and DAAMS.
On the basis of the preliminary results 4 representative sets are selected for further analysis as described above in Section IV.4. The refined results of the KM and ML parameter estimation methods with respect to the WEIBULL core distribution of lifetimes are recorded in Table IV-11 given in Section IV.4.5..
In all cases the goodness of fit is nearly perfect which is demonstrated by the calculation of S-discrepancies recorded in Table IV-10 (Section IV.4.5.) and by the plots of H-residuals as presented in Appendix VII.4.1., pages 1 to 4. Obviously, the KM-method is preferred in all cases, however, the ML-method is also satisfactory. The ML-method applied to M.2.1. gives a 17% higher value of the shape parameter. This deviation is probably caused by the high value of the parameter itself, for then a small inaccuracy in the calculating process can have a relative large effect on the slope of the associated straight line (survivor curve). The ML-method applied to M.6.1. gives a 8.5% higher value of the size parameter. The reason for this deviation may be the small sample size (16 discards). When the number of discards is less than 20, the ML-method applied to the WEIBULL distribution is questionable.
The maximum S-discrepancy related to the core distribution is low in all cases, with the means close to zero and low values for the sums.
The plots of H-residuals are given in Appendices VII.4.1., pages 1 to 4. All these plots are from the KM-method which was preferred here. The fit is quite good in all cases. The plot for Code M.4.1. has some irregularities; this is also indicated by a somewhat lower coefficient of determination (r = 0.985) in comparison with those of M.1.1., M.2.1. and

M.6.1. which have $r > 0.99$. Misspecification was rejected in all cases. Subpopulation I is not seen whereas subpopulation II is found in M.1.1., M.2.1. and M.6.1.. The latter is clearly indicated and encircled in the corresponding variance-stabilized plots of H-residuals related to the core distribution (Appendix VII.4.1., pages 1 to 4). The plots of H-residuals (Appendix VII.2.1., page 1 and Figure 14 for Code M.2.1.) demonstrate a good fit of the curve ($\beta_2 = 1$) to the empirical data. As argued above, the value of $\hat{a}$ has the tendency to be overestimated in comparison with (a) as determined by (21). For that reason the difference $\{\hat{a} - (a)\}$ is calculated. Since according to Table IV-12, $\hat{\mu}_2$ has the tendency to be underestimated in comparison with (μ_2) as determined by (III/5), the directly estimated value of $\hat{H}(1)$ at $t = 1$ is compared with $\{\hat{H}(1)\} = 1/\hat{\mu}_3^2$ which is indirectly estimated. Finally, we examine how far $\hat{t}(1)$ deviates from $t = 1$. The value of $\hat{t}(1)$ is calculated from the integrated hazard at that point in time; then it holds that:

$$\frac{\hat{t}}{\hat{\mu}_2} = \left(\frac{\hat{t}}{\hat{\mu}_3}\right)^2 \qquad \text{(integrated hazard in common at } t = \hat{t}\text{)}$$

Hence it follows that:

$$\hat{t}(1) = \hat{\mu}_3^2/\hat{\mu}_2 \tag{22}$$

The findings for Phase II are summarized below.

CODE	KIND OF EQUIPMENT	$\hat{t}(1)-1$	$\hat{a}-(a)$	$\hat{H}(1)$	$\{\hat{H}(1)\}$
M.1.1.	Milling equipment	0.252	0.395	0.007	0.006
M.2.1.	Lathes	1.202	1.380	0.012	0.005
M.6.1.	Surface-treating equipment	3.419	4.498	0.016	0.004

As can be deduced from the table above, the difference, $\hat{H}(1) - \{\hat{H}(1)\}$, is minor for Code M.1.1. The differences associated with Code M.2.1. and Code M.6.1. for Phase II seem relatively large which is shown by the plots in Appendix VII.4.2., page 2 and 4 respectively. In terms of the probability of survival, the differences are negligible:

Code M.1.1.: $\hat{S}(H=0.007) = 0.993$ against $\hat{S}(H=0.006) = 0.994$

Code M.2.1.: $\hat{S}(H=0.012) = 0.988$ against $\hat{S}(H=0.005) = 0.995$

Code M.6.1.: $\hat{S}(H=0.016) = 0.984$ against $\hat{S}(H=0.004) = 0.996$

How our model represented by Figure 6 in Chapter III fits the empirical retirement data is graphically demonstrated by the plots in

Appendix VII.4.2., pages 1 to 4. The dotted survivor curves represent the modelled part. The plot related to M.1.1. shows a perfect fit with respect to both Phases II and III. One outlier related to Phase II is clearly illustrated by the plot of M.2.1. and of M.6.1.. In both cases Phase II contains only two points, T_1 and T_2, where T_2 is also in Phase III. Since point T_2 (situated close to $\hat{a}$) is generally a more accurate estimate than point T_1 closer to $t = 1$, the curve ($\beta_2 = 1$) is positioned through the point with coordinates $\ln \hat{H}(T_2)$ and $\ln T_2$. The same procedure is adopted for Codes D.6.3. and 3-1. The deviation between the modelled curve and the data points of Phase II seems significant but is negligible in terms of S-discrepancies as determined above.

IV.6. Results for Industrial Equipment (Dalcy)

The 20 sets of industrial capital equipment classified by different functions as recorded in Dalcy (CBS, 1987) are described above in Section IV.2. and listed in Table IV-2 (Appendix VII.1., page 1).

The results of a preliminary graphical analysis followed by segregation of subpopulations and an informal parameter estimation of the WEIBULL core distribution are summarized in Table IV-7 (Appendix VII.3., page 1). A detailed description of these analyses and the plots of empirical data points are given in the M.Sc.-thesis work of HINSKENS & VAN WIERINGEN (1989) directed by BEKKER and DAAMS.

On the basis of preliminary results 7 representative sets are selected for further analysis as described above in Section IV.4. The results of the KM and ML parameter estimation with respect to the WEIBULL core distribution of lifetimes are recorded in Table IV-11 in Section IV.4.5.. In all cases the goodness of fit is evident from the low S-discrepancies in Table IV-10 (Section IV. 4.5.), and the plots of H-residuals in Appendix VII.5.1., pages 1 to 7. Obviously, the KM-method is preferred in 4 cases and the ML-method in the remaining three. Again, the KM-method of parameter estimating meets the criteria of robustness in all cases. The ML-method applied to Codes D.1.1., and D.9.1. results in lower values of the related S-discrepancies. The differences in value of the parameter estimates according to the KM-method on the one hand and the ML-method on the other are negligible.

This category of capital assets is characteristic for heavily aggregated sets of empirical retirement data which is the reason that the shape parameter in two cases, Codes D.5.2. and D.9.1., turns out to be low in value. This seems to be in conflict with the principles of our lifetime distribution model. When $\beta = 2$ is regarded as a lower limit, the β-estimate related to Code D.5.2. is 12% lower, and the one related to Code D.9.1. is 19% lower. The ratio, β/β_{ϕ}, 1.12 and 1.19 respectively, is quite normal on the basis of results obtained by simulation of erroneous data (Section IV.3.1.). Neglecting heterogenity may be considered as equivalent to the adoption of a homogeneous set with measurement errors. Hence, aggregation has the same effect in reducing the value of β. Aggregation is clearly demonstrated by the value of the β-estimate related to Code D.10.2. which is, in fact, an aggregate of several equipment sets listed in Table IV-6 (Appendix VII.3., page 1). As we have seen, the β-estimates of the one kind of equipment are significantly higher in value which can be derived from Table IV-11 above (Codes M.1.1., M.2.1., M.4.1., and M.6.1. as compared with Code D.10.2). The maximum S-discrepancy related to the core distribution is low in all cases. The highest Sdis (0.106) is established for Code D.2.1.. The means are close to zero and the values for the sums are low in all cases. Subpopulation I was not apparent but in 5 sets (Codes D.1.1., D.2.1., D.7.1., D.9.1. and D.10.2.) there are discards recorded at $t = 1$, which is the common point for Phase I and II. Subpopulation II is found in Codes D.2.1., D.6.3., D.7.1. and D.10.2. The latter is clearly indicated and encircled in the corresponding variance-stabilized plots of H-residuals in Appendix VII.5.1., page 1 to 7.

The plots of H-residuals related to Phase II in Appendix VII.2.1., pages 1 and 2 demonstrate a good fit of the curve ($\beta_2 = 1$) to the empirical data points. It appears that in all cases the values of $\hat{a}$ and $\hat{H}(1)$ are overestimated as compared with (a) and $\{\hat{H}(1)\}$. The findings for Phase II are summarized below.

CODE	KIND OF EQUIPMENT	$\hat{t}(1)-1$	$\hat{a}-(a)$	$\hat{H}(1)$	$\{\hat{H}(1)\}$
D. 2.1.	Lorries and trucks	0.674	0.935	0.016	0.010
D. 6.3.	Pumps and compressors	1.457	2.307	0.008	0.003
D. 7.1.	Electric generators	0.778	1.183	0.006	0.004
D.10.2.	Machining equipment	9.507	10.332	0.019	0.002

The differences as shown in the table above are illustrated by the plots in Appendix VII.5.2., pages 1 to 7. In terms of the probability of survival, the differences are negligible as shown below:

Code D. 2.1.: $\hat{S}(H=0.016) = 0.984$ against $\hat{S}(H=0.010) = 0.990$

Code D. 6.3.: $\hat{S}(H=0.008) = 0.992$ against $\hat{S}(H=0.003) = 0.997$

Code D. 7.1.: $\hat{S}(H=0.006) = 0.994$ against $\hat{S}(H=0.004) = 0.996$

Code D.10.2.: $\hat{S}(H=0.019) = 0.981$ against $\hat{S}(H=0.002) = 0.998$

Even in the case of Code D.10.2. the difference in $\hat{S}(1)$ is no more than 0.017 which is low indeed. Misspecification of Phase II was rejected in the cases discussed above.

The refined and formal results per set are discussed below.

Code D.1.1.: Passenger and delivery cars.

Phase II is not met in the preliminary graphical analysis. The first data point at t = 1 could be related to Phases I, II and III (Appendix VII.5.2., page 1). The core distribution fits well which is also seen in the ML plots of the H-residuals (Appendix VII.5.1., page 1).

Code D.2.1.: Lorries and trucks.

Phase II clearly appeared in the modelled plot (Appendix VII.5.2., page 2) although Phase I may be present, as indicated by one data point at t = 1. The plot of H-residuals related to Phase II show that the fit of an EXPONENTIAL distribution to the data points is very good (Appendix VII.2.1., page 1). The overestimate of partition parameter $\hat{a}$ is 0.935 years. This error reduces to 0.261 years if $t = \hat{t}(1)$ instead of t = 1 is taken as the duration of Phase II. The modelled curve related to Phase II is at a somewhat higher probability of survival than the regression line. Apart from that small deviation, our model fits quite well. This is also demonstrated by the KM plots of the H-residuals (Appendix VII.5.1., page 2) with respect to the core distribution.

Code D.5.2.: Wrapping equipment.

Phase I and II are absent. The fit of the modelled WEIBULL core distribution to the data points is very good in spite of the heavily aggregated set of empirical retirement data (Appendix VII.5.2., page 3). The high coefficient of determination (r = 0.997) is probably due to some smoothing effect. The KM and the ML sets of H-residuals are practically identical (Appendix VII.5.1., page 3).

Code D.6.3.: Pumps and compressors.

As shown by the plot of H-residuals in Phase II, the "fit" of an EXPONENTIAL distribution to the two data points is good (Appendix VII.2.1., page 2). The overestimate of partition parameter $\hat{a}$ is 2.307 years. This error reduces to 0.85 years if $t = \hat{t}(1)$ instead of t = 1 is taken as the duration of Phase I. The modelled curve for Phase II is at a somewhat higher probability of survival than the regression line (Appendix VII.5.2., page 4). The fit of the modelled WEIBULL core distribution to the empirical retirement data is very good as demonstrated by the KM set of plots of H-residuals (Appendix VII.5.1., page 4).

Code D.7.1.: Electric generators.

As shown by the plot of H-residuals for Phase II, the fit of an EXPONENTIAL distribution to the data points is quite good (Appendix

VII.2.1., page 2). The overestimate of partition parameter $\hat{a}$ is 1.183 years. This error reduces to 0.405 years if $t = \hat{t}(1)$ instead of $t = 1$ is taken as the duration of Phase I. The modelled curve for Phase II and the regression line are close together. The reasonable fit of our model is shown by the modelled curve (Appendix VII.5.2., page 5). The goodness of fit for the core WEIBULL distribution, is demonstrated by the related KM plots of H-residuals (Appendix VII.5.1., page 5).

Code D.9.1.: Measuring and controlling equipment.

Phases I and II are absent because this is a heavily aggregated set of empirical retirement data. Roughly 5 data points in the right tail of the core distribution have about equal probabilities of survival at different lifetimes (Appendix VII.5.2., page 6). The probability of survival seemed to be time-independent as far as these 5 outliers are concerned. Of course, this is rare and probably due to aggregation. Apart from this, the fit of the WEIBULL core distribution to the remaining empirical retirement data is good. When the plots of H-residuals (Appendix VII.5.1., page 6) are considered, the results using the KM-parameters are better than those from the ML-parameters. Obviously, the KM-method is less sensitive for outliers in the tail of the distribution. Anyhow, this set of empirical retirement data is in fact inappropriate for testing and judging of our model. The ML parameter estimates result in more favourable S-discrepancies.

Code D.10.2.: Machining equipment.

As already stated above, this set is an aggregate of a collection of equipment and tools as recorded in Table IV-1 (Appendix VII.1., page 1). The high coefficient of determination ($r = 0.991$) is probably due to the effect of aggregation. As shown by the plot of H-residuals for Phase II, the "fit" of an EXPONENTIAL distribution to two data points is good (Appendix VII.2.1., page 2). The overestimate of partition parameter $\hat{a}$ is 10.332 years. This large error reduces to 0.825 years if $t = \hat{t}(1)$ instead of $t = 1$ is taken as the duration of Phase I. Consequently, the modelled curve for Phase II is at a higher probability of survival than the regression line (Appendix VII.5.2., page 7). The same plot shows that the fit of the modelled curve of the core WEIBULL distribution to the data points of Phase III is quite good. This is also demonstrated by the KM set of plots of the H-residuals (Appendix VII.5.1., page 7).

IV.7. Modelling of WINFREY Type Curves

WINFREY (1931/1935) studied 176 sets of empirical retirement data representing many varieties of property goods including industrial property. He constructed 18 typical survivor curves consisting of 3 classes which are related to the position of the modal age or lifetime relative to the average life. This classification resulted in:

- 6 Left modal curves which have the mode to the left of average life, denoted as:
  L^0, L^1, L^2, L^3, L^4 and L^5 type curves.
- 7 Symmetrical curves which have the mode approximately at average life, denoted as:
  S^0, S^1, S^2, S^3, S^4, S^5 and S^6 type curves.
- 5 Right model curves which have the mode to the right of average life, denoted as:
  R^1, R^2, R^3, R^4 and R^5 type curves.

WINFREY (1935) has tabulated for these 18 curves the probability of survival versus age in percentages of the average age (= 100%). For this we may refer to the following tables in BULLETIN 125 (1935) of the IOWA Engineering Experiment Station:

- Table 21 (pages 102 and 103): 6 Left modal types
- Table 21 (pages 104 and 105): 7 Symmetrical types
- Table 21 (pages 105 and 106): 5 Right modal types.

The data tabulated for WINFREY's 18 curves have been plotted in a (ln H), (ln t) grid followed by a quasi-linear regression technique in order to estimate the parameters. In this manner a picture is obtained of the extent to which a particular WINFREY type curve differs from a WEIBULL distribution. The results of this conversion are summarized in Table IV-13 on the next page.

It is striking that many of the WINFREY curves do not differ much from WEIBULL survivor curves. The coefficients of determination r are reasonable high. In particular, type curves, L^1, L^2, L^3, S^1, S^2 and S^3, are good approximations of a WEIBULL. Several curves are partly straight or fairly straight up to $\ln H < 1$ and then change gradually to a concave shape. Some curves have a more or less convex curvature although a concave curvature is more in accordance with theory.

WINFREY Type Curve	WEIBULL Estimates			Remarks
	$\hat{\beta}_3$	$\hat{\mu}_3$	r	
L^0	1.668	103.229	0.992	Straight if ln H<1; concave if ln H>1
L^1	1.999	109.492	0.997	Fairly straight if ln H<1; concave if ln H>1
L^2	2.741	115.036	0.999	Nearly straight line
L^3	4.005	117.976	0.987	Somewhat curvature (convex)
L^4	5.657	117.797	0.977	Curvature (convex)
L^5	8.675	118.135	0.941	Straight if ln H<-5; concave if ln H>-5
S^0	1.930	111.218	0.993	Straight if ln H<1; concave if ln H>1
S^1	2.726	109.497	0.998	Straight if ln H<1; concave if ln H>1
S^2	3.990	109.560	0.998	Nearly straight line
S^3	5.245	110.969	0.996	Fairly straight line
S^4	8.914	112.073	0.980	Somewhat curvature (convex)
S^5	13.403	110.661	0.969	Somewhat curvature (convex)
S^6	21.861	108.804	0.954	Somewhat curvature (convex)
R^1	2.659	112.135	0.976	Strong curvature (concave)
R^2	3.099	107.884	0.979	Strong curvature (concave)
R^3	2.901	113.072	0.980	Curvature (concave)
R^4	3.868	113.453	0.975	Curvature (concave) in lower part
R^5	11.430	104.273	0.981	Curvature (convex)

Table IV-13: Results of conversion of WINFREY type curves to WEIBULL survivor curves.

A concave survivor curve, in particular for the part with ln H <-5, is a fair approximation of a 3-component (composite) WEIBULL distribution as in our model. In this respect curves R^3 and R^4 are good approximations. Bearing in mind that a WEIBULL distribution is:

- left modal when $\beta < 3.2$;
- approximately symmetrical when $3.3 < \beta < 3.5$;
- right modal when $\beta > 3.6$,

it can be concluded that the WEIBULL survivor curves derived from WINFREY survivor curves cannot be specified according to WINFREY's classification. This is demonstrated by the values of the shape parameter as specified in Table IV-13 above:

- L-class: $1.668 < \hat{\beta}_3 < 8.675$
- S-class: $1.930 < \hat{\beta}_3 < 21.861$
- R-class: $2.659 < \hat{\beta}_3 < 11.430$

After a closer examination of the WINFREY type curves it appeared that:

- Type L^0, L^1 and L^2 curves have their origin at t = 0 when S(0) = 1, wheras the origins of the remaining L-curves start at:
  L^3: t = 5% of the average life (= 100%)
  L^4: t = 20% of the average life (= 100%)
  L^5: t = 40% of the average life (= 100%)
- Type S^0, S^1 and S^2 curves have their origin at t = 0 when S(0) = 1, whereas the origins of the remaining S-curves start at:
  S^3: t = 10% of the average life (= 100%)
  S^4: t = 30% of the average life (= 100%)
  S^5: t = 45% of the average life (= 100%)
  S^6: t = 60% of the average life (= 100%)
- Type R^1, R^2, R^3 and R^4 curves have their origin at t = 0 when S(0) = 1, whereas the origin of the R^5 curve starts at t = 30% of the average life (= 100%).

In conclusion, WINFREY considered a pre-aging period with a constant probability of survival of 100% whilst constructing his 3 classes of survivor curves; it can be seen that the steeper the curve when converted to a {ln(-ln S)}, (ln t) grid the longer the duration to 100% survival. In that respect WINFREY's curves are rather like our model. However, the principles differ.

The 65 sets of property goods which are documented in BULLETIN 103 (1931) reflect the following WINFREY type curves:

- Type curves L^1, L^2 and L^3 : 24 times (36.9%)
- Type curves S^1, S^2 and S^3 : 13 times (20.0%)
- Type curves R^1, R^2, R^3 and R^4: 18 times (27.7%)

The L^1, L^2, L^3, S^1, S^2 and S^3 curves are reasonable approximations of WEIBULL survivor curves whereas type curves R^1, R^2, R^3 and R^4 are more or less fair approximations of a 2-component WEIBULL model. The R^3 and R^4 type curves are, when converted into a (ln H), (ln t) grid, concave and well approximated by two straight lines, one related to Phase II (β = 1) and one related to Phase III ($\beta \geq 2$). With the result indicated above it can be expected that at least 55 sets out of the 65 documented assets will show a reasonable to good fit when tested on the basis of our model. The results will be discussed in Section IV.7.1..

With respect to the additional 111 sets of property goods which are documented in BULLETIN 125 (1935), the most frequently adopted type curves are L^1, L^2 and L^3 (26.7%), S^1, S^2, S^3 and S^4 (23.9%) and R^1, R^2,

R^3 and R^4 (30.1%). Here again, more than 80% of the documented sets are adapted to WINFREY type curves which are reasonable approximations of WEIBULL models.
In the next subsection 15 representative sets are selected from the 65 sets documented in BULLETIN 103 (1931) for more detailed analyses.

IV.7.1. Results for IOWA Property Goods

The 65 sets of property goods are described and analyzed by WINFREY (1931) and listed in Table IV-3 (Appendix VII.1., page 2). After a closer examination of the kind of property, one may conclude that we are dealing with several groups of the same kind. For instance, poles, cables, lamps, crossties, rolling railway equipment, agricultural tools and automobiles. Some of these groups consist of primitive goods such as poles and crossties.
The results of a preliminary graphical analysis followed by segregation of subpopulations and an informal parameter estimation of the WEIBULL core distribution are summarized in Table IV-8 (Appendix VII.3., page 2). A detailed description of these analyses and the plot of empirical data points are given in the M.Sc.-thesis work of HINSKENS and VAN WIERINGEN (1989) directed by BEKKER and DAAMS.
In view of the recorded groups and on the basis of preliminary results 15 representative sets are selected for further analyses as described above in Section IV.4.. The results of the KM and ML parameter estimation with respect to the WEIBULL core distribution of lifetimes are shown in Table IV-11 which is contained in Section IV.4.5..
In all cases the goodness of fit is evident from the low S-discrepancies given in Table IV-10 (Section IV.4.5.), and the plots of H-residuals (Appendix VII.6.1., pages 1 to 15). Obviously, the KM-method is preferred in 8 cases and the ML-method in the remaining 7 cases. Again, the KM-method of parameter estimation meets the criteria of robustness in all cases. The ML-method applied to Codes 11-2, 14-1, 24-5, 33-1, 34-1, 44-6 and 59-1 resulted in favourable values of the related S-discrepancies. The differences in value of the μ-parameter estimates according to the KM and the ML-method are insignificant (less than 4%). The β-estimates deviate much more, e.g., Codes 14-1 (16%), 30-4 (22%), 44-6 (13%), 53-1 (46%), 56-1 (11%) and 64-11 (11%). These deviations between the KM and ML-estimates of the shape parameters are mainly caused by peculiar empirical retirement data. More on this subject later. The deviation with

respect to Code 44-6 is solely due to a high value of the β-estimate, the highest recorded in Table IV-11.

Some of the WINFREY empirical retirement data involve more or less aggregated sets, e.g., Code 9-1 (Central Office Equipment) for which $\hat{\beta} = 1.659$ is found. That is lower than the expected lower limit of $\beta = 2$ according to the principles of our lifetime model. In this case the β-ratio amounts 2.000/1.659 = 1.206 which is within the range expected when the impact of aggregation/measurement error is taken into calculation. Aggregation can/will be the cause of a higher standard deviation and, consequently, of a lower valued shape parameter.

The maximum S-discrepancy related to the core distribution is sufficiently low to ensure a good fit in all cases. The highest Sdis (-0.167) occurred with Code 53-1 which is due to peculiar empirical retirement data. The means are close to zero and the values of the sums are low in all cases.

Subpopulation I can be seen in the set of Code 64-2 (Automobiles) which is not surprising for assets manufactured in the beginning of this century. Subpopulation II is found in Codes 3-1, 30-4, 38-1, 53-1 and 64-2. Also other data sets may contain suppopulation II as will appear later on. None of the so-called primitive goods seemed to involve subpopulation I or II. It is noted that in those days (1852 - 1930) many assets were restored after a sudden and too early failure because restoration was not as costly as it is now. In the variance-stabilized plots of H-residuals the data points related to Phase II are encircled.

The plots of H-residuals for Phase II (Appendix VII.2.1., pages 3 and 4), demonstrate a good fit of the curve ($\beta_2 = 1$) to the empirical data points except for Code 38-1. It appeared that in 4 cases the values of $\hat{a}$ and $\hat{H}(1)$ are overestimated as compared with (a) and $\{\hat{H}(1)\}$. In one case (Code 38-1) it appeared that $\hat{a}$ and $\hat{H}(1)$ are slightly underestimated. The findings for Phase II are summarized below.

CODE	KIND OF PROPERTY	t(1)-1	$\hat{a}$-(a)	$\hat{H}(1)$	$\{\hat{H}(1)\}$
3-1	Water work pumps	1.683	2.838	0.005	0.002
30-4	Madza B-lamps (60W)	0.043	0.050	0.026	0.025
38-1	Coal flat train cars	-0.872	-0.339	0.002	0.002
53-1	Rodger ballast train cars	1.886	3.371	0.006	0.002
64-2	Automobiles 1900-1922	0.587	0.674	0.031	0.019

As can be deduced from the table above, the difference, $\hat{H}(1) - \{\hat{H}(1)\}$, is zero for Code 38-1, and relatively small for the remaining sets. In terms of the probability of survival, the differences are negligible:

Code 3-1 : $\hat{S}(H=0.005) = 0.995$ against $\hat{S}(H=0.002) = 0.998$
Code 30-4: $\hat{S}(H=0.026) = 0.974$ against $\hat{S}(H=0.025) = 0.976$
Code 38-1: apparently no difference
Code 53-1: $\hat{S}(H=0.006) = 0.999$ against $\hat{S}(H=0.002) = 0.998$
Code 64-2: $\hat{S}(H=0.031) = 0.969$ against $\hat{S}(H=0.019) = 0.981$

Misspecification of Phase II was rejected in all the cases discussed above except Code 38-1. Misspecification of the WEIBULL core distribution was rejected in all cases. How well our model according to Figure 6 (Chapter III) fits the empirical retirement data points is shown by the plots in Appendix VII.6.2., pages 1 to 15.

In addition, plots are shown wherein each complete set of data points is presented with its survivor curve(s) both for our model and the relevant WINFREY curve. The survivor curve(s) according to our model is/are represented by the solid straight line(s). The relevant WINFREY curve is marked with (+) dots. The position of the WINFREY curve is such that the coordinates of the intersection with the WEIBULL survivor curve are:

$$x = \ln \hat{\mu}_3 \quad \text{and} \quad y = \ln \hat{H}(\hat{\mu}_3) = 0$$

This does not necessarily mean that the sum of the S-discrepancies for the relevant WINFREY curve was reduced to a minimum. Nevertheless, a comparison of the goodness of fit can be made. The plots are shown in Appendices VII.6.3., pages 1 to 5.

The formal results for each set of empirical retirement data are discussed below.

Code 3-1: Water works pumps.

The KM plots of H-residuals (Appendix VII.6.1., page 1) indicate 3 peculiar points in the area $0.7 < H < 1$. This can be understood by considering WINFREY's own description: "The majority of the pumps represented were retired because of inadequacy; a few were discarded because they were obsolete and the remainder scrapped. They were installed from 1853 to 1892".

As shown by the plot of H-residuals for Phase II (Appendix VII.2.1., page 3), an EXPONENTIAL distribution fits the data well. The overestimation of partition parameter $\hat{a}$ is 2.838 years. This error reduces to 1.155 years if t = $\hat{t}(1)$ instead of t = 1 is taken as the duration of Phase I. The modelled curve for Phase II is at a somewhat higher probability of survival than the regression line (Appendix VII.6.2., page 1). The fit of the modelled WEIBULL core distribution to the empirical retirement data points is quite good.

WINFREY (1935) found a "good shape of curve" and a good fit to his type curve L^3. According to Table IV-13, that type curve is a good approximation of a WEIBULL survivor curve. Indeed, both curves for Phase III agree significantly as appeared in the relevant plot (Appendix VII.6.3., page 1). The latter plot shows that this WINFREY type curve is unsuitable for Phase II.

Code 4-1: Steam engines.

The KM plots of H-residuals (Appendix VII.6.1., page 2) indicate some peculiar data in the area of $0.6 < H < 1.3$. This can be understood by considering WINFREY'S own description: "In practically all cases the engines could have run for an indefinite period, but the cost of repairs and operation was high, and the requirements of greater capacity and greater economy caused their replacement". Also the number of discards ($d = 17$) is too low for a sound statistical analysis.

There are no data points in Phases I or II; all data points are from Phase III. The good fit of our model to these data points is demonstrated by the plot related to Phase III (Appendix.6.2., page 2).

WINFREY (1935) found a "poor shape of curve" and a fair goodness of fit to his type curve L^4 which is not a good approximation of a WEIBULL survivor curve. This is shown by the relevant plot (Appendix VII.6.3., page 1). Our model agree significantly with the data points.

Code 9-1: Central office equipment.

The KM plots of H-residuals (Appendix VII.6.1., page 3) indicate some peculiarities which are probably due to a heavily aggregated set characterized by a low value of the shape parameter estimate ($\beta < 2$). The high coefficient of determination ($r = 0.995$) is an indication of some smoothing effect. WINFREY (1931) gives no further information on discarding aspects.

There are no data points in Phases I or II; all data points are from Phase III. The good fit of our model to these data points is not only demonstrated by the KM set of plots of H-residuals but also by the plot of modelled data points for Phase III (Appendix VII.6.2., page 3). The S-discrepancies are small.

WINFREY (1931) found a "good shape of curve" and a fair goodness of fit to his type curve L' which is a fair approximation of a WEIBULL survivor curve. The goodness of fit of both the WINFREY curve mentioned and a WEIBULL survivor curve is clearly demonstrated by the relevant plot (Appendix VII.6.3., page 1), and favours our model slightly.

Code 11-2: Aerial cables.

The ML plots of H-residuals (Appendix VII.6.1., page 4) indicate a perfect fit. This can also be concluded on the basis of the Sdis (Table IV-10, Section IV.4.5.). The calculated coefficient of determination r according to the KM-estimation followed by a quasi-linear regression technique is one.

WINFREY (1931) gives the following information: "The cable was used in the exchange and toll service of the New Yersey Division of the New York Telephone Co.. The data cover the removals accounted for up to Jan. 1, 1916". It is noted that the discards of this kind of assets are measured in "units of capital" and not in physical units.

There are no data points in Phases I or II. The perfect fit of our model to the data points of Phase III is not only demonstrated by the ML plots of H-residuals but also by the plot of modelled data points for Phase III (Appendix VII.6.2., page 4).

WINFREY (1935) found a "good shape of curve" and also a good fit to his type curve S'. Not surprisingly, because that type curve is a fair approximation of a WEIBULL survivor curve. Indeed, both curves coincide up to ln H = 1 as is demonstrated by the relevant plot (Appendix VII.6.3., page 2). In the area ln H > 1 the WINFREY curve changes gradually from nearly straight to concave. Although the WINFREY curve results in a good fit, our model fits even better.

Code 14-1: Underground cables.

The ML plots of H-residuals (Appendix VII.6.1., page 5) indicate a regular but slightly curved pattern and a good fit which can also be concluded on the basis of the S-discrepancies (Table IV-10, Section VII.4.5.). WINFREY (1931) gives the following information: "The cable was main cable, exclusive of coarse gauge, and was in the service of the New York Telephone Co. in the Brooklyn and Queens Divisions. The data include the removals accounted up to Jan. 1, 1916". It is noted that the discards of this kind of assets are measured in "units of capital" and not in physical units.

There are no data points in Phases I or II; all data points appear to fall in Phase III. However, when the plot of modelled data points (Appendix VII.6.2., page 5) is examined, one or two data points may belong to Phase II. Furthermore, it is noted that the data points of Phase III suggest a convex curvature which may be an indication for erroneous data as far as the observed lifetimes are concerned. Indeed, after the introduction of a location parameter of -2 years (all T's change into T+2), the curvature disappeared and the new parameter estimates were:

Phase III: $\hat{\beta}_3$ = 3.027; $\hat{\mu}_3$ = 18.123 years; (a) = 4.339 years and r = 0.999

instead of $\hat{\beta}_3$ = 2.379; $\hat{\mu}_3$ = 15.955 years; (a) = 2.140 years and r = 0.996

In conclusion, the shape parameter is increased as well as the coefficient of determination. An additional result of this exercise is that Phase II is more clearly represented by one data point. Herewith we have shown what the impact is of an absolute and constant error in observed lifetimes. Since the correction made above is not based on the given empirical retirement data, it will not be considered in the next. WINFREY (1935) found a "good shape of curve" and a fair goodness of fit to his type curve S^1. Not surprisingly, because that type curve is a fair approximation of a WEIBULL survivor curve. As shown by the relevant plot (Appendix VII.6.3., page 2) both curves coincide in the area where $\ln H < 1$. In the area $\ln H > 1$ the WINFREY curve changes gradually from nearly straight to concave which results in this case in a better fit than achievable with our model.

Code 24-5: Wooden poles.

The ML plots of H-residuals (Appendix VII.6.1., page 6) indicate a regular pattern and a good fit which can also be deduced from the S-residuals (Table IV-10, Section 4.5). WINFREY (1931) gives the following information: "The statistics go back to 1852 and include the experience of the North German and Prussian Telegraph systems. The poles were treated with coal tar". It is noted that these kinds of assets must be regarded as primitive goods.

No data points fall in Phases I or II; all the data seemed to fall in Phase III. However, when the plot of modelled data points (Appendix VII.6.2., page 6) is examined, 2 or 3 data points may fall in Phase II. The maximum Sdis for Phase II turned out to be -0.0075 which is very low indeed. The good fit of our model to the data points is evident.

WINFREY (1935) found a "good shape of curve" and also a good fit to his type curve S^2 which is according to Table IV-13 (Section IV.7.) a good approximation of a WEIBULL survivor curve. As shown by the relevant plot (Appendix VII.6.3., page 2) the WINFREY curve is too steep for a good fit. Our model fits better.

Code 30-4: Mazda B-lamps (60W).

The KM plots of H-residuals (Appendix VII.6.1., page 7) indicate a regular pattern and a good fit which can also be deduced on the basis of the S-discrepancies (Table IV-10, Section IV.4.5.). The plot of variance-stabilized H-residuals suggests that 2 data points are from Phase II whereas Phase I is not represented. WINFREY (1931) gives the following information: "The life experience of 100, 60-Watt, Mazda B lamps as found by the Engineering Department of the National Lamp Works, Cleveland, Ohio, and published in their Bulletin No.13 F, pp. 7-9, Dec. 5, 1917".

As shown by the plot of H-residuals for Phase II (Appendix VII.2.1., page 3), the fit of an EXPONENTIAL distribution to the data points is very good. The overestimation of partition parameter $\hat{a}$ is 0.05 years and thus negligible.

The plot according to our model (Appendix VII.6.2., page 7) demonstrates a very good fit to the data points for both Phases II and III.

WINFREY (1935) found a fair "shape of curve" and a fair goodness of fit to his curve type R^2 which is according to Table IV-13 (Section IV.7.) not a good approximation of a WEIBULL survivor curve but is a reasonable approximation of our model. This is demonstrated by the relevant plot (Appendix VII.6.3., page 3). The WINFREY curve fits not too badly, but our model does better.

The introduction of a location parameter of only -1 year (all T's change to T+1) has the following effects:

Phase III: $\hat{\beta}_3 = 4.472$; $\hat{\mu}_3 = 7.417$ years; (a) = 4.164 years and r = 0.997

instead of $\hat{\beta}_3 = 3.763$; $\hat{\mu}_3 = 6.387$ years; (a) = 3.265 years and r = 0.995.

By this small adjustment of observed lifetimes the shape parameter estimate related to Phase III increased and the coefficient of determination improved. Herewith it is again demonstrated what the effect of a small absolute error in observed lifetimes can have on the estimates of the WEIBULL parameters and, consequently, on the proof of validity of our model elaborated in Chapter III..

Code 33-1: Steam locomotives.

The ML plots of H-residuals indicate a regular but somewhat wavy pattern. A good fit is suggested on the basis of the S-discrepancies (Table IV-10, Section IV.4.5.). WINFREY (1931) gives the following information: "The experience of 781 steam locomotives on the U.P.R.R., C.B. and Q.R.R., and C.R.I. and P.Ry as compiled by E.J. Kates, engineer on the staff of the Nebraska State Railway Commission, Nov. 10, 1910".

No data points fall in Phases I or II; all the data are from Phase III. The plot for our model (Appendix VII.6.2., page 8) demonstrates a good fit to the data points.

WINFREY (1935) found a good "shape of curve" and also a good fit to his type curve L^3 which is a fair approximation of a WEIBULL survivor curve. The relevant plot (Appendix VII.6.3., page 3) illustrates that both survivor curves agree well and demonstrate a good fit. The data points suggest a somewhat convex curvature in line with curve L^3. However, the question is whether or not the observed lifetimes are sufficient accurate. After the introduction of a location parameter of -2 years (all T's change into T+2), the curvature nearly disappeared and the new parameter estimates were:

Phase III : $\hat{\beta}_3$ = 4.252; $\hat{\mu}_3$ = 30.900 years; (a) = 10.760 years and

r = 0.992

instead of: $\hat{\beta}_3$ = 3.996; $\hat{\mu}_3$ = 28.797 years; (a) = 9.382 years and

r = 0.991

By this small adjustment of the observed lifetimes, the shape parameter increased and the coefficient of determination improved.

Code 34-1: Passenger train cars.

The ML plots of H-residuals (Appendix VII.6.1., page 9) indicate a regular but somewhat wavy pattern. A good fit is suggested on the basis of the S-discrepancies (Table IV-10, Section IV.4.5.). WINFREY gives the following information: "The life experience of several thousand passenger cars, on the U.P.R.R., C.B. and Q.R.R., and C.R.I. and P.Ry. as compiled by E.J. Kates, engineer on the staff of the Nebraska Railway Commission, Nov. 10, 1910".

No data points fall in Phases I or II. The good fit of our model for Phase III data is evident as shown in Appendix VII.6.2., page 9.

WINFREY (1935) found a "good shape of curve" and a fair goodness of fit to his type curve S^3 which is a fair approximation of a WEIBULL survivor curve. The relevant plot (Appendix VII.6.3., page 3) illustrates that the WINFREY type curve fits well in the area ln H > -3 but rather badly in the area ln H < -3. Again, our model fits better and agrees significantly with the data points.

Code 38-1: Coal flat train cars.

The KM plots of H-residuals (Appendix VII.6.1., page 10) indicate a regular pattern and a nearly perfect fit to Phases II and III which is confirmed by the determination of the S-discrepancies (Table IV-10, Section 4.5.). WINFREY (1931) gives the following information: "The experience of 2,712 coal flat cars on the U.R.R., C.B. and Q.R.R., and C.R.I. and P.Ry. as compiled by E.J. Kates, engineer on the staff of the Nebraska State Railway Commission, Nov. 10, 1910. The vacations were caused as follows: 13.3% change of service, 36.2% worn out, 28.3% accidental destruction, 4.4% sold, 17.8% still in service". Here the censoring is evident but the effect is negligible since the number of discards is several thousands. Discards due to accidental obstruction (28.3%) is in accordance with those for Phase II.

There are no data points in Phase I whereas 8 data points fall in Phase II. The plot for our model (Appendix VII.6.2., page 10) demonstrates the nearly perfect fit of the data points for Phase III. The data points for Phase II fit well also to the modelled survivor curve. As shown by the plot of H-residuals related to Phase II (Appendix VII.2.1., page 3), the fit of an EXPONENTIAL distribution to 6 data points is good, and to 3 (one in common with Phase III) data points is less good. The reason is that the slope of the Phase II curve is set on $\beta_2 = 1$ but the slope of a regression line through these points turned out to be $\hat{\beta}_2 = 1.557$ with $r = 0.993$. In fact, the EXPONENTIAL distribution seemed in this case a misspecification if the empirical retirement data were accurate. Phase II could be either a mixture of change failures and wear and tear failures or the data are inaccurate. Anyhow, the difference with the modelled survivor curve is not great.

WINFREY (1935) found a "good shape of curve" and a reasonable goodness of fit to his type curve R^3 which is a fair approximation of our lifetime model. As demonstrated by the relevant plot (Appendix VII.6.3., page 4) the WINFREY type curve is concave; it fits well in the area ln H > -2, and is not too bad in the area ln H < -2 that includes also Phase II. Again, our model fits quite well to both Phases II and III.

Code 44-6: Railway cross ties.

The ML plots of the H-residuals (Appendix VII.6.1., page 11) indicate a regular pattern and suggests quite a good fit which can also be concluded on the basis of the S-discrepancies related to Phase III. WINFREY (1931) gives the following information: "The life experiences of 43,681 cross ties of Douglas fir species when treated with zinc chloride. They were set in 1901 at various places on the Southern Pacific System and subjected to heavy traffic. Data were collected by the Forest Products Laboratory, Madison, Wis.."

Of course, crossties are primitive goods whose replacement will occur as soon as they fail to carry out their task due to physical decay. When the first symptoms of unacceptable decay are found, all cross ties set in the same period at long railway tracks will be replaced. Consequently, the discarding process occurs relatively fast and may be the cause of a steep survivor curve. Indeed, all 14 sets of crossties (Table IV-8, Appendix VII.3., page 2) have high shape parameter values. The highest is found for Code 41-3 (informal β-estimate turned out to be 14.771).

There are no data points in Phases I or II. Phase I is not expected here but Phase II may be possible due to train-derailments. The plot for our model (Appendix VII.6.2., page 11) shows that the fit to the points in Phase III is almost perfect.

The difference between the shape parameter estimates obtained by the KM-method and ML-method may be reduced by the introduction of a location parameter of -1 year (all T's change to T+1). The result of this operation is:

Phase III: $\hat{\beta}_3$ = 6.986; $\hat{\mu}_3$ = 12.818 years; (a) = 8.371 years and r = 0.992

instead of $\hat{\beta}_3$ = 6.990; $\hat{\mu}_3$ = 11.749 years; (a) = 7.787 years and r = 0.988

The effect on the shape parameter is insignificant but the coefficient of determination improved.

WINFREY (1935) found a good "shape of curve" and a fair goodness of fit to his type curve R^4. The real fit is demonstrated by the relevant plot (Appendix VII.6.3., page 4) and appeared to be poor. The WEIBULL survivor curve according to our model agrees quite well with the data points.

Code 53-1: Rodger ballast train cars.

The KM plots of H-residuals (Appendix VII.6.1., page 12) indicate an irregular pattern and many outliers due to peculiar data. The maximum Sdis (-0.167) is the highest found so far. However, this maximum

discrepancy is less than 0.20 which is the threshold value specified for a good fit. The maximum Sdis related to Phase III is lower (0.096) when the parameters obtained by the ML-method were applied. However, the mean of the discrepancies is not close to zero and the sum is much higher.
WINFREY (1931) gives the following information: "The life experience of 760 Rodger ballast cars placed in service 1892 to 1897. The data were obtained from protestant's exibit No.71, before the Interstate Commerce Commission, Valuation Docket No. 327, Great Northern Railway Company and Montana Eastern Railway Company, 1923".
Phase I does not appear. As shown by the plot of H-residuals related to Phase II (Appendix VII.2.1., page 4), the fit of an EXPONENTIAL distribution to the data points is very good. The overestimation of partition parameter $\hat{a}$ is 3.371 years. This error reduces to 1.485 years if $t = \hat{t}(1)$ instead of $t = 1$ is taken as the duration of Phase I. The modelled curve related to Phase II is at a higher probability of survival than the regression line (Appendix VII.6.2., page 12).
In spite of the peculiar data, the results for Phase III can be improved by the introduction of a location parameter of -3 years (all T's change to T+3) which has the following effects:
Phase III: $\hat{\beta}_3 = 4.160$; $\hat{\mu}_3 = 25.003$ years; (a) = 9.028 years, and

$r = 0.979$.
instead of $\hat{\beta}_3 = 3.477$; $\hat{\mu}_3 = 21.976$ years; (a) = 6.311 years and

$r = 0.974$.
The shape parameter estimate related to Phase III increased to a value which is much closer to the one found on the basis of the ML-method. Although the coefficient of determination improved slightly, the fit of the core distribution to the data points is not as good as in the foregoing cases. This is also shown by the modelled plot in Appendix VII.6.2., page 12. Nevertheless, misspecification of our model to the data points is rejected.
WINFREY (1935) found a good "shape of curve" and a poor goodness of fit to his type curve R^4. In view of the peculiar data the poor fit may be clear and is in line with our findings so far. However, curve R^4 is a reasonable approximation of our model; the fit is only good in the area $\ln H > -0.8$, and extremely poor for Phase II as demonstrated by the relevant plot (Appendix VII.6.3., page 4). In spite of the peculiar data, our model performs better.

Code 56-1: Corn cultivators (1-row).

The KM plots of H-residuals (Appendix VII.6.1., page 13) indicate no regular pattern due to somewhat peculiar data. Generally, a good fit is suggested which is also evident on the basis of the S-discrepancies related to Phase III. WINFREY (1931) gives the following information: "The life experience of 56, 1-row corn cultivators used in Hardin Country, Iowa, 1875 to 1924. They were used 7 to 30 days a year. Most of the machines were housed. Data compiled by Iowa Engineering Experiment Station, 1924".

No data points fall in Phases I or II. However, in the plot of modelled data points there is one data point that may be from Phase II. Our lifetime model fits quite well to all data points as shown in Appendix VII.6.2., page 13.

WINFREY (1935) found a good "shape of curve" and a good fit to his type curve L^3. Not surprisingly, because that curve is a reasonable approximation of a WEIBULL survivor curve. The relevant plot (Appendix VII.6.3., page 5) demonstrates that both curves agree well with the data. Nevertheless, the survivor curve related to our model fits even better, particularly, in the lower area.

Code 59-1: Grain binders (5 to 8-foot)

The ML plots of H-residuals (Appendix VII.6.1., page 14) indicate no regular pattern due to somewhat peculiar data. Nevertheless, a good fit is suggested by the H-residuals as well as by the S-discrepancies in Phase III. WINFREY (1931) gives the following information: "The life experience of 45, 5- to 8-foot, grain binders used in Hardin Country, Iowa, 1882 to 1924. They were used 3 to 24 days a year. Most of the machines were housed. Data compiled by the Iowa Engineering Experiment Station, 1924".

No data points appear in Phases I or II. The plot of modelled data points of Phase III demonstrates quite a good fit to our model as shown in Appendix VII.6.2., page 14.

WINFREY (1935) found a fair "shape of curve" and a good fit to his type curve L^2 which is a good approximation of a WEIBULL survivor curve. The relevant plot (Appendix VII.6.3., page 5) demonstrates a reasonably fit to the L^2 type curve and quite a good fit to our lifetime model. The relevant WINFREY type curve is a bit too steep.

Code 64-2: Automobiles 1900-1922.

The KM plots of H-residuals (Appendix VII.6.1., page 15) indicate a regular pattern and suggests quite a good fit of our model to the data points related to Phase III. The plot of variance-stabilized H-residuals shows clearly 4 data points of Phases I and II. WINFREY (1931) gives the following information: "The life experience of 3,124 automobiles (1,028 model T-Ford and 2,096 other makes) registered in 12 representative Iowa countries for 1922 and reported to the State during the year as being dismantled or otherwise removed from service. Information collected by the Iowa Engineering Experiment Station, 1925".

Phase I is represented by one data point related to a probability of survival of $\hat{S}(0.5) = 0.98367$ derived from the KM-estimate which is associated with an integrated hazard of:

$$\hat{H}(0.5) = -\ln \hat{S}(0.5) = 0.01646$$

According to our model:

$$\hat{H}(0.5) = \left(\frac{t}{\hat{\mu}_1}\right)^{\hat{\beta}_1} = \frac{t^{\hat{\beta}_1}}{\hat{\mu}_3^2} = \frac{0.5^{\hat{\beta}_1}}{7.196^2} = 0.01646$$

Then it follows that $\hat{\beta}_1 = 0.23$ which is indeed a realistic value since $0 < \beta_1 < 1$. As shown by the plot of H-residuals related to Phase II (Appendix VII.2.1., page 4), the fit of an EXPONENTIAL distribution to the data points is good. The overestimation of partition parameter $\hat{a}$ is 0.674 years. This small error reduces to 0.087 years if $t = \hat{t}(1)$ instead of $t = 1$ is taken as the duration of Phase I. The modelled curve for Phase II is at a somewhat higher probability of survival than the regression line. This is shown by the plot in Appendix VII.6.2., page 15, which demonstrates a good fit of our model to the data points.

WINFREY (1935) found a good "shape of curve" and a good fit to his type curve S^2 which is a good approximation of a WEIBULL survivor curve. The relevant plot (Appendix VII.6.3., page 5) demonstrates that the S^2 type curve fits the data points in Phase III well but poorly to those in Phase II. Our model fits well to Phases I and II and very well to Phase III and thus is superior here.

This case is an interesting one, not only because Phase I was present, but we have another set of automobiles, i.e., passenger cars (Code P.C.NL) discarded in 1977 which is discussed and compared in the next section.

IV.8. Results for Miscellaneous Sets

The sets to be considered are described above in Section IV.2. and listed in Table IV-4 (Appendix VII.1., page 1). The results of a preliminary graphical analysis followed by segregation of subpopulations, if any, and an informal parameter estimation for the WEIBULL core distribution are summarized in Table IV-9 (Appendix VII.3., page 1).

The refined results of the KM and ML parameter estimation methods with respect to the WEIBULL core distribution of lifetimes are recorded in Table IV-11 which is inserted in Section IV.4.5..

In all cases the goodness of fit is evident as demonstrated by the calculation of the S-discrepancies recorded in Table IV-10 (Section IV.4.5.), and by the plots of H-residuals as represented in Appendix VII.7.1., pages 1 to 4. Obviously the KM-method is preferred for dwellings and passenger cars, whereas the ML-method is better for bus tyres. Again, the KM-method followed by a quasi-linear regression technique in estimating the parameters meets the criteria of robustness in all cases. It is noted that the set of empirical retirement data for dwellings has been analyzed and discussed by BEKKER (1980). The histogrammes are given on the next page. The set related to Code D.NL48 was subject to the more refined parameter estimation methods discussed in Section IV.4. and thereafter. The results are practically the same as far as the KM-method is concerned:

Informal: $\hat{\beta}_3$ = 3.65 ; $\hat{\mu}_3$ = 93.18 years

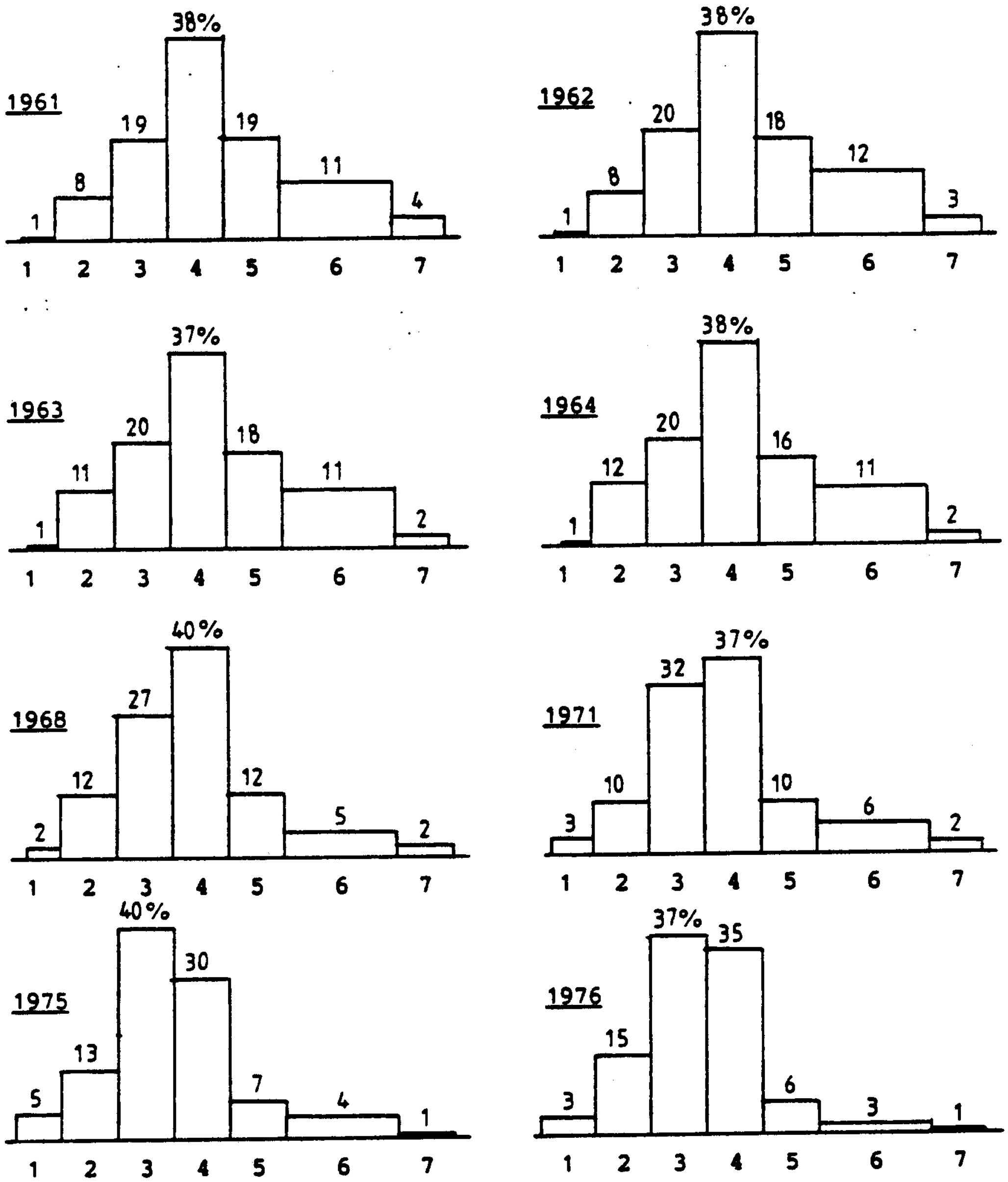
Refined : $\hat{\beta}_3$ = 3.547; $\hat{\mu}_3$ = 93.46 years

The results of the ML-method deviate (shape parameter 16% less, size parameter 3% more than estimated by the KM-method).

The set related to Code D.NL12 refers to the same retired assets from the dwelling stock, however, the empirical retirement data are grouped in the following 14 age classes (in years):

<11 years	86 - 89
11 - 14	93 - 101
18 - 26	111 - 114
36 - 39	118 - 126
43 - 51	161 - 164
61 - 64	168 - 176
68 - 76	>176 years

The age group of 14 years and younger for Phase II is disregarded because of accidental causes. Since a reasonable deal of old dwellings are



Age Classes of Dwelling		
Age Class at Discarding	Added to the Dwelling Stock	Age Range (years) as per Class
7	before 1801	176 years or more
6	1801 - 1850	126 - 175
5	1851 - 1875	101 - 125
4	1876 - 1900	76 - 100
3	1901 - 1925	51 - 75
2	1926 - 1950	26 - 50
1	1951 - 1976	25 years or less

Age class histograms of dwellings as withdrawn from the dwelling stock (Netherlands) in 8 distinctive calendar years. Source: BEKKER (1980) classified as monuments, the hazardous process differs from that related to Phase III. For this reason age groups of 161 years and older are segregated. The remaining 9 age classes are related to Phase III. As a consequence of data grouping the discrepancies in empirical observations will be smoothed out to some extent and the standard deviation of the core distribution will decrease. Indeed, the WEIBULL shape parameter estimate increased from 3.547 (Code D.NL48) to 3.881 (Code D.NL12) which is roughly 10%. Also the size parameter increased from 93.469 (Code D.NL48) to 102.266 (Code D.NL12), recorded in Table IV-11 as a result of the KM-estimation method. The latter is due to a time-scaling effect because the discarding observations are recorded in 8 distinct years (1961, 1962, 1963, 1964, 1968, 1971, 1975 and 1976) within a time span of 15 years. Consequently, the time scale is enlarged by 7 to 8 years which may be regarded as a kind of location parameter. This can also be derived from the age classes as tabulated above, which are discontinuous. It is noted that the differences in value of the parameter estimates according to the KM-method on the one hand and the ML-method on the other are significant. This is probably due to the effect of large age classwidths (25 to 50 years) and grouping of data on a discontinuous time scale. For that purpose the ML-method is inappropriate. For the same reason the H-residual plotting as applied to the sets of data sources 1, 2 and 3, is questionable and, statistically speaking, incorrect. Nevertheless, it would be acceptable in the case of dwellings to use the same uniform method but the results must be interpreted with great care; they can/will differ from the results obtained by the observation of individuals whose lifetimes are measured in calendar years and a classwidth of one year is used in the statistical analysis.

As far as the two other miscellaneous sets (Codes P.C.NL and B.T.NL) are concerned, the differences between the parameter estimates obtained by the KM or ML-method are insignificant. Both sets are concerned with observations of individuals.

Phase I was not represented in the empirical retirement data. Phase II was found for passenger cars. Since this data set may be regarded as sufficiently accurate, there is almost no difference between the empirical findings and our model (to be discussed later).

The refined and formal results per set of empirical retirement data are discussed below.

Code D.NL12: Dwellings (grouped) NL.

The KM plots of H-residuals (Appendix VII.7.1., page 1) indicate a regular pattern and a quite good fit of the curves to the data points with the exception of two outliers. The latter data points are related to very old dwellings which may be subject to a different hazard process associated with the conservation of objects of historical interest. One data point is thought to be from Phase II but it is not revealed as such by the plot of variance-stabilized data points. Thus the assumption that this particular data point represents Phase II seems questionable.

The plot of modelled data points (Appendix VII.7.2., page 1) demonstrates quite a good fit to our model. The outliers from the oldest age classes are clearly seen. The modelled survivor curve for Phase II agrees with the one and only data point which may or may not represents subpopulation II in the set of empirical retirement data.

As stated above, the results may be interpreted with great care unless an adequate test confirms the goodness of fit (see Table IV-14 in Section IV.9).

Code D.NL48: Dwellings NL.

The KM plots of H-residuals (Appendix VII.7.1., page 2) indicate a somewhat irregular pattern which may be caused by the large classwidths (25 to 50 years) and the joining of empirical retirement data of 8 distinct years within an observation time span of 15 years. This irregular pattern nearly disappears in the variance-stabilized plot except for the 8 data points related to the oldest age classes. The latter deviation, which may be the consequence of a different hazardous mechanism, is enlarged in the plot {-ln S(t)} versus {H(t)}.

The plot of modelled data points (Appendix VII.7.2., page 2) demonstrates quite a good fit of our model to the data points. The 8 data points for the oldest age classes are clearly seen as outliers which may be subject to a different hazardous mechanism. The 4 data points in the youngest age class are also outliers which may represent Phase II. The latter is supported by a low valued maximum S-discrepancy (0.00027) as compared with the modelled survivor curve for Phase II. As found previously by BEKKER (1980), misspecification of the model applied to this set of discarded dwellings is rejected.

As stated above, the results may be interpreted with great care unless an adequate test confirms the goodness of fit (see Table IV-14 in Section IV.9).

Code P.C.NL: Passenger cars NL.

The KM plots of H-residuals (Appendix VII.7.1., page 3) indicate regular patterns and quite a good fit which is ascertained by the results obtained by the analysis of S-discrepancies related to Phases II and III. The 4 encircled data points from Phase II can be seen in the plot of variance-stabilized H-residuals.

One data point at t = 1 year may be related to both Phases I and II. Phase II is clearly represented by 4 data points. The findings for Phase II are summarized below.

CODE	KIND OF CAPITAL ASSETS	$\hat{t}(1)-1$	$\hat{a}-(a)$	$\hat{H}(1)$	$\{\hat{H}(1)\}$
P.C.NL.	Passenger cars NL	0.105	0.145	0.012	0.011

All differences, $\hat{H}(1) - \{\hat{H}(1)\}$, $\hat{a} - (a)$, and $\hat{t}(1) - 1$, are very small. As shown by the plot of H-residuals related to Phase II (Appendix VII. 2.1., page 4), the fit of an EXPONENTIAL distribution to the data points is nearly perfect. The overestimation of partition parameter $\hat{a}$ is only 0.145 years. This small error reduces to 0.04 years if $t = \hat{t}(1)$ instead of t = 1 is taken as the duration of Phase I. The modelled curve for Phase II is at a slightly higher probability of survival level than the regression line in the plot which represents our model (Appendix VII.7.2., page 3). The fit of the core distribution to the data points is also very good. Only one data point which represents the oldest age class deviates somewhat from the expected value but this discrepancy is insignificant and probably the consequence of inaccurate observations (old cars never die!).

When the life characteristics of these passenger cars discarded from stock in 1977 are compared with the discards recorded in 1925 by WINFREY (1931), the similarity is striking. Both sets (P.C.NL and 64-2) contain coincidently 14 data points of which 4 data points are from Phase II and the remaining 10 from Phase III. It can be ascertained that during a development period of more than 50 years the size parameter increased from 7.2. to 9.4 years which leads to a probability of survival at t = 1 year of 0.9809 and 0.9888 respectively. The age at which cars made in 1900-1913 and cars made 50-60 years later have equal survival probabilities is when the two integrated hazards are equal.

That is at t, where:

$$(t/7.196)^{3.365} = (t/9.416)^{4.389}$$

Then it follows that $t \approx 22.8$ years. No doubt, much progress is made in the automobile industry which is expressed by a, technically speaking, more reliable product, safer at high speed, with greater comfort and, moreover, a longer lifetime when the lifetime is expressed not in years but as distance.

Lorries and trucks (Code D.2.1.) belong to the same family with a rolling transportation function. Not surprisingly, the life characteristics as demonstrated by the plot (Appendix VII.5.2., page 2) are very like passenger cars (Codes 64-2 and P.C.NL). Again, the fit of our model applied to this kind of capital assets is excellent.

Code B.T.NL: Bus tyres.

The ML plots of H-residuals (Appendix VII.7.1., page 4) indicate regular patterns and a nearly perfect fit which is ascertained by the results obtained by the analysis of S-discrepancies related to Phase III.

No data points fall in Phases I or II. However, bus tyres are subject to accidents which mean that one of the components of a composite distribution must represent a time-independent hazardous mechanism. Obviously, the number of observations (264 individual tyres) was too small because the probability of survival at $t = \hat{a}$ (140,092 km) at the end of Phase II, is very high: $\hat{S}(a) = 0.999$. This is demonstrated by the plot for our model (Appendix VII.7.2., page 4). The nearly perfect fit is evident.

One may argue that tyres are primitive goods and not capital assets as such. When passenger cars, lorries and trucks are regarded as capital assets, we must realize that "rolling" tyres are a substantial part of the capital stock. Anyhow, tyres are interesting objects for life studies. Since they can be retreaded or not, the replacement decision is an economic one as in the case of capital assets or manufactured durables in service. The life characteristics of bus tyres represent the principles of our lifetime model which is demonstrated above.

IV.9. Summary and Conclusions

In this chapter our probabilistic lifetime model elaborated in Chapter III was tested. For that purpose 96 sets of empirical retirement data have been employed. These sets refer to 4 main sources among which are the original 65 sets documented in Bulletin 103 by WINFREY (1931). Almost all sets of empirical retirement data are more or less aggregated groups or categories or kind of capital assets and manufactured durables. The problem of heterogeneity and inaccurate measurement was solved by assuming a homogeneous mass of which empirical retirement data of individuals are erroneously measured and recorded. The degree of data dispersion due to measurement errors was simulated in order to gather insight into the sensitivety of our model to these problems. The simulation was based on a hypothetical WEIBULL distribution with distinct sets of parameters. Then the probability of survival, $S(T_i)$, at every point in time, T_i, is known. It was assumed that T_i is erroneously measured as $T_{i.\phi}$ where the $T_{i.\phi}$'s are normally distributed with mean T_i and a spread which was determined by a specified coefficient of variation (constant per simulation run for each T_i). Hence, the deviation between $T_{i.\phi}$ and T_i increases with the value of T_i and in reverse. Then the simulated value of $T_{i.\phi}$ was paired with $S(T_i)$ for i = 1,000 erroneously measured lifetimes. These 1,000 data points were used to determine the parameter estimates by means of a quasi-linear regression technique. The findings of this error simulation process are:

- The generated erroneous data points remain WEIBULL distributed; the goodness of fit is significant unless the coefficient of variation with respect to lifetime measurement errors is > 0.10.
- The shape parameter of a WEIBULL distribution decreases as the coefficient of variation (of the normally distributed error term applied to the lifetime variable) increases. This result is predictable because the coefficient of variation of a WEIBULL distribution depends solely on its shape parameter. The coefficient of variation related to the hypothetical WEIBULL distribution will be enlarged by the generation of erroneous lifetimes and, consequently, the WEIBULL distribution that fits the erroneous data points has a lower valued

shape parameter. A reduction of 0 to 25% and even more is possible when the population mass is heavily aggregated and/or measuring and recording of discards is inaccurate. The reduction of the shape parameter value is less when the size parameter value increases.

- The size parameter will change with respect to simulated measurement errors. Although the change in value is positively related to the coefficient of variation, that change can result in a lower or higher valued size parameter. Higher when μ_3 = 10 years and β_3 > 2, μ_3 > 25 years and β_3 > 4. Lower when μ_3 > 25 years and $\beta_3 \approx 6$ or more. For a given degree of inaccuracy the change in value of the size parameter is much less and negligible in comparison with the change in value of the shape parameter. Of course, this is logical but with the aid of simulation techniques the changes are quantified.

In the light of the above we have demonstrated that the shape parameter estimate of the heavily aggregated population related to Code D.10.2. is significantly lower than those of the less aggregated populations corresponding to Codes M.1.1. to M.7.1. (mechanically operated tools in an engineering works). Another example is Code D.1.1. ($\hat{\beta}_3$ = 2.31) which is a more aggregated database of passenger and delivery cars than Code P.C.NL. ($\hat{\beta}_3$ = 4.389) which is a database of passenger cars only.

An absolute error in the recorded lifetimes can easily be eliminated by the introduction of a location parameter. Those kinds of errors are exposed by curvature in a WEIBULL plot. A convex curvature can be changed to a (nearly) straight line by a negative location parameter (in units of time); a concave curvature was not met. This adjustment was made for 5 sets documented by WINFREY (1931/1935) simply to demonstrate the effect. It was found that a small location parameter of -1 to -3 years can be sufficient to improve the goodness of fit.

A comprehensive testing procedure was applied consisting of the following steps:

1. Determination of the nonparametric estimate of the integrated hazard $\hat{H}(T_j)$ as developed by KAPLAN and MEIER (1958).
2. Plotting of data points in a $\{\ln \hat{H}(T_j)\}$ versus $(\ln T_j)$ grid.
3. Graphical decomposition of subpopulations, if any.

4. Estimation of WEIBULL parameters of each component of the composite distribution following two different methods:
   . Quasi-linear regression technique (KM-method)
   . Maximum likelihood technique (ML-method)
5. Plotting of 2 sets of 4 graphs which represent the integrated hazard residuals, one set based on the parameter estimates obtained by the KM-method (quasi-linear regression), and one based on the parameter estimates obtained by the ML-method. One of the graphs representing the variance-stabilized integrated hazard residuals is used as an additional tool in segregation of data points into subpopulations, if any.

   The main purpose of these plots was to check misspecification of the core WEIBULL distribution and its parameters. These plots were also used to produce more refined parameter estimates.
6. Calculation of the probability of survival discrepancies (mean, sum and maximum discrepancy) of the core WEIBULL distribution in order to decide upon:
   . which of the parameter estimation methods performs better, and
   . what is the goodness of fit of our model to the data points from Phase III.
7. Fitting of an EXPONENTIAL distribution to the data points from Phase II which appeared in 13 of the 30 representative and selected sets of retirement data. Checking of misspecification by means of H-residual plots of Phase II data points.
8. Estimation of partition parameter, a, in two ways:
   . indirectly from the parameter estimates of the core WEIBULL distribution according to the principle of our model: (a), and
   . directly as the intercept of the regression lines of Phases II and III: $\hat{a}$.

   In addition, the duration of Phase I is estimated and the integrated hazard at that point in time is determined.
9. Plotting of the modelled survivor curves applied to Phase II and III for 30 representative and selected sets. The empirical retirement data points are also plotted to demonstrate graphically the goodness of fit. Comparing of the goodness of fit of our modelled survivor curves with the relevant WINFREY type curves.
10. Evaluation of testing results and estimates including a Chi-square test.

Many of the sets of empirical retirement data show only Phase III. Phase II is found in 13 out of the 30 selected sets. Phase I is represented by only one data point related to Code 64-2 (Automobiles, 1900-1922).
The fit of the modelled survivor curve to Phase III data points is obvious in all cases: nearly perfect in 13 cases, good in 14 cases and fair in 3 cases based on the following maximum values of the probability of survival discrepancies:

- Nearly perfect: less than 0.050 (5.0%),
- Good : 0.051 to 0.100 (5.10 to 10.00%),
- Fair : 0.101 to 0.200 (10.1 to 20.00%),

in combination with a mean as close as possible to zero and a low sum of discrepancies. In the literature the fit is regarded as sufficient when the coefficient of determination obtained by the quasi-linear regression technique is higher than r = 0.95 and the maximum value of the probability of survival discrepancy less than 0.20 (20%). The lowest coefficient of determination (r = 0.974) was established for Code 53-1 that has also the highest Sdis (-0.167). This is due to peculiar data as was established by WINFREY (1931) and re-established in the study ahead.
The goodness of fit of an EXPONENTIAL distribution to the data points in Phase II was tested for 13 empirical retirement data sets. It was found that the size parameter estimate was, generally speaking, underestimated when compared with the value of the size parameter calculated on the basis of our model. The partition parameter, a, was somewhat overestimated but that error disappeared when the estimated duration of Phase I was taken into account. Generally, the duration of Phase I was overestimated when compared with our model. The differences in probability of survival terms were negliglible as compared with the values obtained on the basis of our model. Misspecification of an EXPONENTIAL distribution of Phase II data points was rejected, perhaps with the exception of Code 38-1. The latter set is the only one which showed an underestimated partition parameter and an underestimated duration of Phase I.

The fit of our modelled survivor curves to the empirical retirement data points is clearly excellent. Evidence is given that our modelled survivor curves are better than those constructed by WINFREY (1931/1935), however,

in several cases his curves coincide more or less with ours. It was established that several WINFREY curves (L- and S-type) are good to fair approximations of a WEIBULL distribution whereas some R-type curves are fair approximations of a two or 3-component (composite) distribution.
It was found that parameter estimation by means of the KM-method followed by a quasi-linear regression technique is relative efficient for the case under consideration.

Since empirical data may be grouped into classes, the model fit is finally tested by the Chi-square method. According to KENDALL & STUART (1979) the relative efficiency of Chi-square will tend to zero as the number of observations, n, increases. Since we have many sets with $n > 200$ up to more than 40,000, this test has its limits. The Chi-square test is thus applied only to the core (Phase III) WEIBULL distribution in the model as far as is possible and meaningful. The amount of empirical data related to Phases I and II is insufficient. Besides, both Phases I and II are the left tails of distributions and the Chi-square test is not effective in the tails.
The results of the Chi-square test for the WEIBULL core distribution carried out with the aid of STATGRAPHICS, Version 4.0, are summarized in Table IV-14 on the next page. The significance levels need no further explanation and lead to the acceptance of the hypothesis in all cases considered with the exception of Code 53-1 (Rodger ballast train cars) due to peculiar empirical retirement data for Phase III.

To conclude, the fit of our probabilistic lifetime model on the basis of a 3-component (composite) WEIBULL distribution is demonstrated. Misspecification of our model is rejected on the basis of the specified testing criteria.

CODE	KIND OF CAPITAL ASSET	Parameter Estimates		Estimation Method	Chi-square Sig. Level	Remarks
		$\hat{\beta}_3$	$\hat{\eta}_3$			
M. 1.1.	Milling equipment	5.078	13.498	ML/SG	0.6821	0.4855 for KM; n = 81
M. 2.1.	Lathes	7.647	13.929	SG	0.5092	0.0578 for KM; n = 82
M. 4.1.	Grinding equipment	3.293	13.934	KM	impossible	sample too small
M. 6.1.	Surface-treating equipment	2.904	16.676	KM	impossible	sample too small
D. 1.1.	Passenger and delivery cars	2.307	6.910	ML/SG/KM	0.1143	0.1061 for KM
D. 2.1.	Lorries and trucks	3.266	10.207	KM	0.0242	0.0165 for ML; n = 100
D. 5.2.	Wrapping equipment	1.790	13.159	KM/SG/ML	0.6655	0.6415 for ML/SG
D. 6.3.	Pumps and compressors	2.765	17.709	KM	0.3670	0.3226 for ML/SG; n = 43
D. 7.1.	Electric generators	2.560	16.836	ML/SG/KM	0.4736	0.4555 for KM; n = 149
D. 9.1.	Measuring and controlling equipment	1.904	17.160	KM	0.0570	0.0033 for SG
D.10.2.	Machining equipment	2.844	24.821	SG	0.9481	0.3471 for KM; n = 40
3-1	Water work pumps	2.513	24.496	ML/SG	0.4612	0.3276 for KM n = 49
4-1	Water works steam engines	3.847	33.717	KM	impossible	sample too small
9-1	Central office equipment(telephone)	1.659	9.767	KM	meaningless	heavily aggregated
11-2	Aerial cables (telephone)	2.575	10.902	KM	0.7202	0.6834 for ML/SG
14-1	Underground cables (telephone)	2.765	15.940	SG/ML	0.1541	0.0028 for KM
24-5	Wooden poles (telegraph)	2.842	12.542	SG	0.7459	0.1032 for KM; n = 313
30-4	Mazda B-lamps (60W electric)	4.597	6.613	ML/SG	0.6480	0.2097 for KM; n = 94
33-1	Steam locomotives (rail road)	3.504	28.133	SG	0.1766	0.0229 for ML
34-1	Passenger train cars (rail road)	4.31	36.17	ML	0.1553	0.0190 for KM; n = 200
38-1	Coal flat cars (rail road)	4.85	21.27	ML	0.5952	0.5271 for KM; n = 194
44-6	Crossties (rail road)	6.21	11.80	ML	0.0616	0.0024 for KM; n = 200
53-1	Rodger ballast cars (rail road)	5.06	22.38	ML	-	bad fit
56-1	Corn cultivators (1-row)	3.218	14.425	KM	0.2665	0.1189 for ML
59-1	Grain binders (5 to 8-foot)	2.419	16.381	SG/ML	0.7060	0.6207 for KM
64-2	Passenger automobiles (1922)	3.598	7.410	SG	0.1877	0.0910 for KM; n = 187
D.NL12	Dwellings NL (12 points)	5.000	97.266	ML	0.1332	0.0844 for SG; n = 566
D.NL63	Dwellings NL (discarded in 1963)	3.547	93.469	KM	0.2447	0.1315 for SG; n = 91
P.C.NL	Passenger cars NL	5.223	9.232	SG	0.3123	0.0149 for KM; n = 194
B.T.NL	Bus tyres (The Hague, NL)	3.046	140,629	ML/SG	0.2992	0.1348 for KM

Table IV-14: Results of the Chi-square test for the WEIBULL core distribution in our model (Phases I and II data points are ignored).

Abbrevations for Parameter Estimation Method:
KM = KAPLAN-MEIER + quasi-linear regression technique
ML = Maximum Likelihood
SG = STATGRAPHICS estimates

CHAPTER V

UNIVERSAL DEPRECIATION METHODOLOGY

V.1. Introduction

REDFERN (1955), among others, came to the conclusion that the principal problem in applying a depreciation method arises from lack of knowledge about the length of life of the assets being depreciated. That is only one dimension of depreciation; the other one is the pattern of depreciation. For many purposes, for instance to estimate the national stock of fixed capital, more often than not a straight line depreciation method is employed, so that if an asset is expected to provide for 20 years, it is deemed to be consumed at a steady rate of 5 percent per annum.

BARNA (1957/1959/1961) came to the conclusion that the value of assets declines with age partly because the expectation of further life declines and partly because of increasing maintenance costs. He also introduced the concept of a stochastic lifetime variable. At the same time he subjoined the remark that such a life function is, unfortunately, unknown. If the concept of maintenance is not limited to repairs and restoration but extented to cover our definition given in Section II.3.1., BARNA's point of departure in developing a depreciation concept is in line with our two-dimensional approach:

- a stochastic lifetime variable, and
- the pattern of depreciation associated with the failure pattern.

Productive reproducible capital assets as well as manufactured durables in service are all depreciated in one way or another. This is a means of expressing that from a certain point in time onwards the net or capital value gradually decreases over the course of time due to technical wear and tear and/or economic obsolescence. At the micro-economic level there are many different methods of depreciation to give administrative expression to the decline in value. The method is often chosen arbitrarily, and can vary from degressive to progressive depreciation, in all possible variations and combinations. However it is calculated or determined, a financial provision to compensate for the drop in value will have to be included in the income earned from supplying products or services, since productive investments have to be paid off. If the (capital) value falls below a given limit, the service life terminates. Then the "weakest link in the NPV-chain" breaks which suggests a WEIBULL related hazard function.

In theoretical economics the value of capital is determined by the future income which it is expected to yield. This concept is too general for our purpose because yields depend not only on capital but also on labour (quantitively as well as qualitatively) and many other factors. Competitive performance is far more decisive for discarding as discussed in Section II.2. in terms of the performance rate. It appears that disruptions in performance of reproducible capital assets and manufactured durables in service

- set a depreciation process in motion <u>and</u>
- generate a service life function of a stochastic nature.

In this respect depreciation is related to service life; disruptions in productive performance in the broadest sense are the common underlying factor. The pattern of the depreciation, which can be derived from the disruptions referred to here, would be universal in nature, in the sense that the same pattern obtains both at the micro-economic level (irrespective of what the records show) and at the macro-economic level for determining the capital value of the national (and international) capital stock.

The next part of this chapter is devoted to developing such a universal depreciation methodology and examining its implications. In Section V.2. definitions are given and the approach is amplified. In Section V.3. the mathematical concept of depreciation in relation to our model is described. The implications of the integrated hazard based model are discussed in Section V.3.1. followed by the graphical presentation in Section V.3.2.. It appeared that the size parameter of the core WEIBULL distribution of lifetime plays a crucial role in depreciation as it does in the average capital consumption (per unit of time). Section V.4. is devoted to the size parameter determinants such as initial productivity and technological progress. A comparison of our depreciation model with two other relevant depreciation models is made in Section V.5.. One of these models was developed by the US DEPARTMENT OF LABOR & BUREAU OF LABOR STATISTICS (1979) and is discussed in Section V.5.1.. The other is a pure maintenance costs and planning model for road pavements which is conceptually not unlike our approach towards depreciation. This road pavement maintenance control model is discussed in Section V.5.2.. Finally in Section V.6. an amortization/depreciation model is elaborated. This model is related to the rental price of capital services.

V.2. Definitions and Approach

In accounting practice depreciation is defined as the expression of the decrease in value of a certain means of production over a given period by means of administrative accounting. The accent in that case is thus clearly on the administrative aspect and particularly on book value.
In the literature on capital stock, depreciation is defined as the difference between the actual purchase price (of a new or identical asset) and the capital value. In simple terms, the depreciation is the current difference between the gross and net value, whereby the net value (capital value) is determined on the basis of "productive performance". This last factor is equivalent to productivity as discussed in Section II.2.. Productivity reflects the ratio of total output to input in money value during a defined period. Market prices (including interest rates) are thereby taken into account. In the literature it is assumed that the real productivity of capital assets decreases as a function of time due to use on the one hand and to technological progress on the other. In the meantime we know that the latter applies only to Phase III. During Phase I the real productivity improves gradually and the real productivity in Phase II is approximately constant. In any case, depreciation must be in line with real productivity, or in other words, depreciation is a function of real productivity, and therefore of the changes over time in the ratio of output to input in money value. The U.S. BUREAU OF LABOR STATISTICS (1979) defines depreciation as "the change in productiveness of an asset over time". This definition matches up with our approach. LIPSEY, STEINER & PURVIS (1984) define depreciation simply as the "capital consumption allowance, required to maintain the existing capital stock intact". They equate "capital consumption" with "replacement investment", which is logical and consistent, if one assumes that production is maintained in the most economic manner. There is an important dynamic aspect in this, for it is only possible to maintain stock "intact" if the desired productivity condition is constantly satisfied. If not, the service life of the asset comes to an end.
Some authors derive the depreciation from the price on the relevant second-hand market. CRAMER (1958), for instance, found the exponential curve a satisfactory description of second-hand motor-car prices. In some isolated cases this might be correct but it scarcely need to be pointed

out that this gives rise to fictitious depreciation. For almost all reproducible and depreciable capital assets and for many manufactured durables the second-hand market is narrow and volatile. There are large fluctuations in price on the second-hand market due to the state of that market and the economy. In a depression period prices are particularly low, and in a boom period there is little or no reason for discarding in response to capacity demand. The residual market value does of course play a role in the on-going process of determining the total amount to be depreciated from purchase to withdrawal. It is noted that the residual value can also be negative due to dismantling, demolition and site clearance costs.

From a capital stock point of view it can be said that capital consumption arises from the requirement of maintaining the stock permanently intact, assuming that production and/or services are maintained in the most economic manner. Consequently, depreciation of the existing stock is equal to capital consumption. This accords with the views of several authors, for instance, GRIFFIN (1975) who defined capital consumption as an estimate of the amount of fixed capital "used up" in current production (and service). Net stock is accumulated capital expenditure less capital consumption (depreciation).

For our purposes depreciation could be defined as a provision to compensate for the decrease in value as a consequence of declining productivity in the course of the service life of a reproducible and depreciable capital asset or manufactured durable product in use, whereby the value is determined on the basis of its productive or competitive performance, i.e., of its performance-rate pattern over time.

The resulting mathematical concept of our universal depreciation methodology is given in the next.

V.3. <u>Mathematics of the Depreciation Function</u>

As shown in Chapters II and III the service life can be well represented by means of a 3-component (composite) WEIBULL distribution, which relates to the following life phases:

- <u>Phase I:</u> Start-up and commissioning Period.

  Completely new capital assets such as plant and production equipment, and also certain (newly-developed) manufactured durables often suffer pre-operational setbacks of a technical or operational nature. As these early and initial problems are systematically solved, and as the sale of the manufactured (new) products or the (new) services gains momentum, the performance rate improves. As long as this is the case, the real productivity increases to the required level. The increase in real productivity during this particular period should, theoretically, result in increasing value, i.e., in negative depreciation. On balance, however, the productivity during this period is such that the inputs (sacrifices) may be higher than the outputs (benefits). This may imply capital consumption which can be regarded as an additional investment to be written off during the depreciation period. For this reason the initial investment is taken as the amount needed to cover also the start-up expenditure and pre-operational losses where applicable. In the purchase prices to be paid for standard capital goods such as motor vehicles, excavators, road building equipment, computers, etc., the costs associated with R & D, testing and improvement, are also included. It will become clear in the course of this section, that depreciation also applies to Phase I.

- <u>Phase II</u>: Stable Period.

  The productive potential is at its maximum level. The process of improvement started in Phase I continues but the effect becomes marginal as it is superseded by change failures. The performance rate is now at its highest level, i.e., performance is affected only by accidental "shocks". During this period a straight line depreciation is logical, as we will show in the course of this section.

- <u>Phase III</u>: Decay Period.

  As a result of technical wear and tear and economic obsolescence the performance rate declines over time. The "life" of some capital assets is now in danger; thus the probability of retirement/replacement progressively increases. This process can be further reinforced if the product or service life-cycle is in the post-operational/use period and there are no alternative applications for the means of production or service. Obviously, the net value decreases progressively during Phase III and, consequently, depreciation increases accordingly.

The process of maintaining the existing capital stock intact in the most economic manner can be regarded as a response to the disruption process (failure process) which affects performance. If so, the value is affected, and hence the fall in value, or in other words, the failure process governs capital consumption. Each time a performance failure occurs, it is as if a fraction of the capital stock dies and the value decreases by that fraction.

It is postulated that each fraction of loss in value, regardless of its cause, corresponds to the hazard rate at that point in time or in production or service. By integrating the hazard-rate function associated with our probabilistic lifetime model, the ratio of the cumulative depreciation (capital consumption) and the amount to be depreciated can be represented by the following basic formula:

$$D(t)/C(t) = H(t) \tag{1}$$

where:

$D(t)$ = cumulative depreciation (capital consumption) as a function of lifetime variabele t

$C(t)$ = amount to be depreciated (in constant prices) as a function of lifetime variable t

$H(t)$ = integrated hazard function

In Section II.7. it was found that the amount to be depreciated, $C(t)$, is embodied in the integrated hazard:

$$H(t) = \ln\{C(t)/I\} \tag{II/37}$$

where (I) is the initial investment (purchase price plus Phase I expenditure) possibly corrected by a positive or negative residual value. The cumulative depreciation function, $D(t)$, can be derived by combining (1) and (II/37). Then it follows that:

$$D(t) = I\{H(t).\exp[H(t)]\} \tag{2}$$

From (2) it can be derived that $D(0)$ is zero. When the term between brackets is one, $D(t) = I$. Since this term is monotonically increasing with time, the cumulative amount to be depreciated can/will exceed the value of I. This is in accordance with our definitions of maintenance, capital consumption and depreciation which are generally the same. As stated in Section II.3.1. with reference to maintenance, the issue at stake is an economic (financial) provision that serves as a counterbalance to compensate for a decreasing probability of survival. It is stressed that the maintenance requirement for complete compensation or elimination of decay, loss of performance, etc., may lag behind the level required as soon as the characteristic point in the lifetime is passed. Full maintenance then becomes too expensive, resulting in a decreasing competitive performance. Hence, $C(t) = I/S(t)$ may in part be a fictitious amount to be depreciated by the user(s) of capital equipment, but $C(t)$ is a real amount in determining capital consumption in the broadest sense.

When the ratio [D/C] is applied to each of the three successive and distinctive life phases, formula (1) proceeds to:

$$\{[D/C](t)\} = \int_0^1 h_1(t|\mu_1,\beta_1)dt + \int_1^a h_2(t|\mu_2,\beta_2)dt + \int_a^\infty h_3(t|\mu_3,\beta_3)dt \qquad (3)$$

where:

$h(t)$ = hazard-rate function that generates a WEIBULL distribution of lifetime variable t

μ_1,β_1 = parameters of Phase I for:

$0 < t < 1;\ \mu_1 > 0$ and $0 < \beta_1 < 1$

μ_2,β_2 = parameters of Phase II for:

$1 < t < a;\ \mu_2 > 0$ and $\beta_2 = 1$

μ_3,β_3 = parameters of Phase III for:

$a < t < \infty;\ \mu_3 > 0$ and $\beta_3 > 2$

Thus formula (3) yields:

$$\{[D/C](t)\} = \left(\frac{1}{\mu_1}\right)^{\beta_1} + \left(\frac{a}{\mu_2}\right) - \left(\frac{1}{\mu_2}\right) + \left(\frac{t}{\mu_3}\right)^{\beta_3} - \left(\frac{a}{\mu_3}\right)^{\beta_3} \qquad (4)$$

In Section III.4.1. the following parametric relationships are obtained:

$$\left(\frac{1}{\mu_1}\right)^{\beta_1} = \frac{1}{\mu_2} \quad \text{,and:} \quad \left(\frac{a}{\mu_2}\right) = \left(\frac{a}{\mu_3}\right)^{\beta_3}$$

If we substitute the above into (4) and restrict depreciation to Phase III, we obtain:

$$\{[D/C](t|\mu_3,\beta_3)\} = \left(\frac{t}{\mu_3}\right)^{\beta_3} \qquad \text{for } t > a > 0 \qquad (5)$$

which is the cumulative depreciation (capital consumption) ratio for the core distribution of our probabilistic lifetime model.

On closer examination of (3) it can be concluded that the ratio during:

- Phase I with a duration of one unit of time amounts to:

$$\{[D/C]_{I}(t|\mu_1,\beta_1 \text{ or } \mu_3\} = (t/\mu_1)^{\beta_1} = t^{\beta_1}/\mu_3{}^2 \quad \text{for } 0 < t < 1 \qquad (6)$$

- Phase II has a linear increasing pattern with:

$$\{[D/C]_{II}(t|\mu_2 \text{ or } \mu_3,a)\} = t/\mu_2 = t/\mu_3{}^2 \qquad \text{for } 1 < t < a \qquad (7)$$

- Phase III has a progressively increasing pattern with:

$$\{[D/C]_{III}(t|\mu_3 \text{ or } \mu_2,a,\beta_3)\} = (t/\mu_3)^{\beta_3} = (t/\mu_2^{\frac{1}{2}})^{\beta_3} \qquad \text{for } t > a \qquad (8)$$

Considering the above, the cumulative depreciation ratio function may be generalized as follows:

$$[D/C](t|\mu_3,p,q) = t^{p}/\mu_3^{q} \qquad (9)$$

with: $p = \beta_1$ and $q = 2$ for Phase I

$p = 1$ and $q = 2$ for Phase II

$p = \beta_3$ and $q = \beta_3$ for Phase III

Basic formula (1) can be converted by inserting ESTEBAN's elasticity function for a WEIBULL distribution (II/27) derived in Section II.5.:

$\pi_W(t) = \beta\{1 - [D(t)/C(t)]\}$ and, hence:

$$D(t)/C(t) = 1 - (1/\beta)\pi_W(t) \qquad (10)$$

By means of (10) we have demonstrated that the ratio, D(t)/C(t), is directly related to the rate of change in the interval (t, t+dt) of the probability density function of lifetimes.

Next we introduce the term "net value ratio" written as:

$$1 - \frac{D(t)}{C(t)} = \frac{C(t) - D(t)}{C(t)} = 1 - H(t) \qquad (11)$$

Substituting this term in ESTEBAN'S elasticity function for a WEIBULL p.d.f., we have:

$$\pi_W(t)/\beta = \frac{C(t) - D(t)}{C(t)} = \{1 - \ln[C(t)/I]\} \qquad (12)$$

Above it is shown that the "net value ratio" is proportional to ESTEBAN's elasticity for a WEIBULL distribution, and inversely proportional to the shape parameter of that distribution. The inverse shape parameter was obtained in Section II.6. above:

$$H_W(t^*) = 1/\beta \qquad (II/31)$$

Hence, the "net value ratio" can also be written as:

$$\{1 - \ln[C(t)/I]\} = H_w(t^*).\pi_w(t) \tag{13}$$

The <u>crucial role of the integrated hazard at the characteristic lifetime</u> is evident. Probably, functions (10 to 13) are the most interesting findings in this study. They describe the relationships between the rate of change in capital value and the rate of change in the interval (t, t+dt) of the p.d.f. of lifetimes and show the connection with the underlying discarding process that generates a lifetime distribution. It appears that $H_w(t^*)$ can be reasonably regarded as an elasticity of capital which is further elaborated in Section VI.2.1..

V.3.1. Implications of the Depreciation Model

It is easy to determine when the initial (historical) investment I is fully written off; then D(t) = I. According to (2) it follows that:

$$H(t).\exp[H(t)] = 1 \quad \text{and, thus:}$$

$$H(t) = \exp[-H(t)] = S(t) \qquad \text{for } D(t) = I \tag{14}$$

The solution of (14) for a WEIBULL distribution is:

$$H_w(t) = S_w(t) = 0.56714$$

For the range of β_3 = 2, 3, 4, 5, 6 and 10 the associated (t/μ_3)-quantities are presented in Table V-1 below.

β_3	t/μ_3	β_3	t/μ_3
2	0.753	5	0.893
3	0.828	6	0.909
4	0.868	10	0.945

Table V-1: Quantities (t/μ_3) for various β_3-values on the basis of formula (14)

When the quantities in this table are compared with those associated with the average lifetime as recorded in Table II-1 in Section II.4.2. we find that they are slightly lower (7.8% less for β_3= 3, and 2.1% for β_3= 6).

This implies that the initial (historical) investment is fully written-off shortly before the average lifetime. In practical terms, this agrees with the results of operative depreciation methodologies as applied by fiscal authorities and accountants. The result obtained above also follows when (1), (II/20) and (II/37) are combined. Then we have:

$$D(\bar{t})/C(\bar{t}) = [\Gamma\{1 + (1/\beta)\}]^{\beta} \approx 0.9^{\beta} \quad \text{and,} \tag{15}$$

$$C(\bar{t})/I = \exp[\Gamma\{1 + (1/\beta)\}]^{\beta} \approx \exp[0.9^{\beta}] \approx 2 \tag{16}$$

Accordingly, the above ratios can be determined for the mode, the median, the characteristic lifetime and for the size parameter. The results are:

Mode: $t = \hat{t}$, according to (II/23):

$$D(\hat{t})/C(\hat{t}) = 1 - (1/\beta) \tag{17}$$

$$C(\hat{t})/I = \exp[1 - (1/\beta)] \tag{18}$$

Median: $t = \mathring{t}$, according to (II/24):

$$D(\mathring{t})/C(\mathring{t}) = \ln 2 \tag{19}$$

$$C(\mathring{t})/I = \exp[\ln 2] = 2 \tag{20}$$

Characteristic lifetime: $t = t^*$, according to (II/30):

$$D(t^*)/C(t^*) = 1/\beta \tag{21}$$

$$C(t^*)/I = \exp[1/\beta] \tag{22}$$

Note that (21) is identical to (17), and (22) identical to (18) when $\beta = 2$. (23)

Size parameter: $t = \mu_3$, according to (II/28):

$$D(\mu_3)/C(\mu_3) = 1 \tag{24}$$

$$C(\mu_3)/I = \exp[1] = 2.7183 \tag{25}$$

When $t = \mu_3$, ESTEBAN's elasticity function for a WEIBULL distribution is zero. This is a crucial point in time because the cumulative depreciation ratio is one, whereas the cumulative capital consumption amounts to 2.7183 I. Note that this amount includes $M(\mu_3)$ which is a provision to maintain a Defender at the performance rate level of a Challenger at every point in time.

A Challenger is defined as the newest capital asset or industrial product that embodies the newest technology. Technological progress may be regarded as the main cause for discarding due to economic obsolescence. Every time an innovation in manufacturing takes place, the lifetime of the existing state of the art in technology terminates. This innovative occurence can be regarded as a highly localized event in a time (or manufacturing) continuum, t, which is characteristic for a stochastic point process. In this respect technological progress may be regarded as an integer counting process resulting from learning, practising and creative searching. If the number of innovations related to that integer counting process are distributed according to a non-homogeneous POISSON distribution, the probability of j events in time interval (0,t) is represented by:

$$Pr\{N(t)-N(0)=j\} = \frac{\exp(-y).y^j}{j!} \qquad \text{for } 0 < y \tag{26}$$

where:

$Pr(j)$ = probability of j innovative occurrences in technology, such that the existing state of the art terminates

$y \quad = \int_0^t h(x)dx = H(t)$ = integrated hazard at $x = t$

$h(x)$ = hazard rate related to the innovation process (R & D) as a function of x

The existing state of the art in technology will terminate when $j = 1$ (one fatal innovative occurrence). Then it follows from (26) that:

$$Pr(j=1) = y.\exp(-y) \qquad \text{for } y = H(t) \tag{27}$$

$Pr(j=1)$ attains its maximum when the first derivative of (27) with respect to the single independent variable $y = H(t)$ is zero:

$$\frac{d}{dy}\{Pr(j=1)\} = -\exp(-y).y + \exp(-y) = 0$$

Then it follows that $y = H(t) = 1$

Assuming that $H(t)$ is the WEIBULL hazard rate at a given point $x = t$ in time, $H_w(t)$ is equal to one only when $t = \mu$.

Evidently, $H(\mu) = 1$, is a <u>crucial</u> value in the light of technological progress because the probability of an innovative occurrence, $Pr(j=1)$, attains its maximum value at point μ in time. At the same point in time the probability of survival of the existing state of the art in technology is known, namely $S(\mu) = \exp[-1] = 0.368$. Consequently, point μ_3 in time is when the ratio $D(\mu_3)/C(\mu_3) = 1$.

It is stressed that the duration of the depreciation period with reference to the initial investment I alone is much shorter than $t = \mu_3$.

Furthermore, the duration of the depreciation period is not equal to the service lifespan while the cumulated amount to be depreciated is more than I at every point in time except when $t = 0$ and thus $M(0) = 0$. Note that capital assets such as manufacturing systems are not necessarily retired when fully written off and antiquated, if there is a temporary shortage of capacity, or if funds are too limited to finance replacement and external funding proves difficult. They are more rapidly replaced, whether or not written-off, as soon as favourable alternatives come within reach or when specific circumstances occur, such as the manufacture of new products.

Subsequently, it is interesting to compare the cumulative amount to be depreciated at the characteristic lifetime when the "average (capital) consumption rate" is reduced to a minimum, and at the point in time when the probability of survival is reduced to exp[-1] at $t = \mu_3$. Then we obtain the following ratio:

$$\frac{C(t=\mu_3)}{C(t=t^*)} = \frac{I.\exp[1]}{I.\exp[1/\beta_3]} = \exp[1 - (1/\beta_3)] \tag{28}$$

When "mode" and "median" of the core WEIBULL lifetime distribution are equal, we have according to (III/31), $\beta_3 = \beta^*[3] = 1/(1 - \ln 2)$.

Hence, this case leads to:

$$\frac{C(t=\mu_3)}{C(t=t^*)} = \exp[\ln 2] = 2$$

The ratio according to (28) becomes higher than 2 as β_3 increases and lower than 2 as β_3 decreases. However, the impact of the shape parameter on this ratio is not great (2.30 for $\beta_3 = 6$ and 1.65 for $\beta_3 = 2$).

Herewith the main implications of our theoretical depreciation (capital consumption) model are examined and argued.

V.3.2. Graphical Presentation of the Depreciation Model

According to (1) the cumulative depreciation (capital consumption) ratio is identical to the integrated hazard at any point in the service life. Consequently, the depreciation (capital consumption) pattern corresponds to Figure 5 (Section III.4.3.) representing the 3-component integrated hazard plot for three successive and distinctive life phases.

Figure 7 on the next page is a graphical representation of the model concerned. The curves are for $\beta_3 = 2, 3, 4, 5$ and 6 resulting in different patterns. The curve for $\beta_3 = 2$ has no linear part because it represents the limiting case when Phase II is absent and, hence, $a = 0$. The curve for $\beta_3 = 4$ is complete; the intersection of the linear and the concave part at $t = a$ is indicated. Phase I is invisible due to the scale $0 < t = 1$.

The remaining curves ($\beta_3 = 3, 5$ and 6) correspond to their respective Phase III's only.

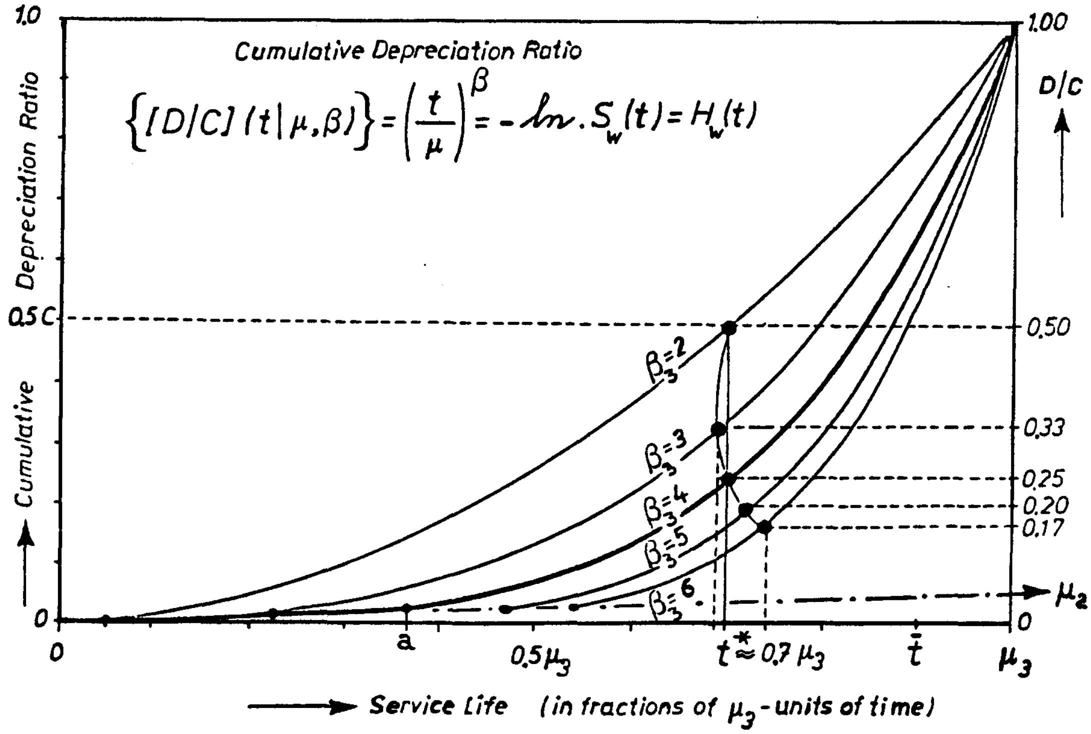


Fig. 15: Cumulative depreciation ratio patterns with different shape parameters β_3 and their common size parameter μ_3.

Figure 15 also indicates that the various Phase III curves have a common Phase II curve, since they have μ_3 in common and thus $\mu_2 = \mu_3^2$. This implies that the various intersections at t = a are situated on their common linear Phase II curve which is characteristic of a straight line depreciation ratio when $\beta_2 = 1$.

We note that the pattern of the curves in Figure 15 correspond to the one in Figure 5 (Section III.4.3.) which implies that each component of the 3-component curve starts from the origin at $(t/\mu_3) = 0$ and $t = 0$ respectively.

In Figure 15 the point on each curve at the characteristic lifetime is another notable feature which was discussed in more detail in Sections II.6. and V.3.1.. It appeared that the cumulative depreciation ratio at $t = t^* \approx 0.7\ \mu_3$ is inversely proportional only to the shape parameter of the core lifetime distributions at a probability of survival of $S_w(t^*) = \exp[-1/\beta_3]$.

As shown by Figure 15, the cumulative depreciation ratio at $t = t^*$ decreases as β_3 increases. Note that the "average total capital consumption" at $t = t^*$ is scarcely affected by the value of β_3, whereas the size parameter μ_3 is determinant for a given I as argued in Section II.6. and elsewhere.
Since $t = \mu_3$ is a crucial issue in the context of our depreciation and capital consumption model as also shown by Figure 15, attention is focussed on the size parameter in the next section.

To conclude, the ambition of BARNA (1957) and many others to achieve universal depreciation patterns which coincide with their failure pattern, is now realized. This applies also to BIØRN, HOLMØY & OLSEN (1989) who discussed gross and net capital in Norway. For that work they employed four (wrongly) assumed survival functions. Not surprinsingly, they came to the conclusion that empirical evidence on survival profiles is strongly needed for further econometric work on the subject of how to measure real capital stocks and flows of capital.

V.4. Size Parameter Determinants

Following DE LA MARE (1982), a firm will continue operating an existing asset until its operating costs equated the market price because it would maximize the net present value (NPV) of that asset. The point of equilibrium is attained when:

$$P_0.\exp[(a - b)t_r] = C_0.\exp[(a + c)t_r] \tag{29}$$

where (in original notations):

P_0 = unit price paid for a product at time t = 0

C_0 = initial operating cost at time t = 0

a = general rate of cost inflation p.a. which, it is assumed, applied equally well to prices as costs

b = rate of technological progress p.a.

c = rate of decline in production efficiency as the manufacturing asset becomes older p.a.

t = time since a plant was first installed and commissioned

t_r = optimal discarding age (in years).

From (29) it follows that:

$$t_r = \frac{\ln(P_0/C_0)}{b + c} \quad \text{,where } P_0/C_0 = \text{initial productivity} \tag{30}$$

In (30) the initial productivity holds for the productivity which is achieved after commissioning. According to our definition, the initial productivity holds at t = 1 when Phase I is passed and when the performance rate is maximized. Furthermore, the optimal discarding age is not affected by inflation if inflation applies equally to prices and costs. The rate of decline in efficiency in (30) is not the same as defined in Section II.2.. In DE LA MARE's view a decline in efficiency means that the operating costs of older equipment increase with time as compared with identical new equipment. That is the reason why (30) contains an additional term representing technological progress. The sum, b + c, in (30) may be interpreted as the rate of decline in performance (efficiency x effectiveness) p.a. as we have defined in Section II.2. above.

Assuming that the optimal discarding age coincides on average with the point in time when an asset is fully depreciated, we obtain:

$$\bar{\mu}_3 \approx t_r = \frac{\ln(P_0/C_0)}{b + c} \tag{31}$$

where $\bar{\mu}_3$ is the expected (average) size parameter.

Herewith we have made plausible that the expected size parameter of the core WEIBULL distribution of lifetimes is proportional to the logarithm of the initial productivity, and inversely proportional to the rate of decline in performance p.a. when $\bar{\mu}_3$ is expressed in years.

Realistic values can be obtained from (31), e.g. when:

$$P_0/C_0 = 1.65; \text{ and } (b + c) = 0.04,$$

it follows that $\bar{\mu}_3$ = 12.5 years.

At the optimal age the net present value amounts: (32)

$$NPV(t_r) = P_0 \cdot {}_0\int^{t_r} \exp[-(r + b - a)t]dt - C_0 \cdot {}_0\int^{t_r} \exp[-(r - a - c)t]dt - I$$

where r is the nominal discount interest rate in a discount factor of, $\exp[-r.t]$.

By integration of (32) we obtain the following net present value up to time (t_r):

$$NPV(t_r) = P_0 \frac{1 - \exp[-(r + b - a)t_r]}{(r + b - a)} - C_0 \frac{1 - \exp[-(r - a - c)t_r]}{(r - a - c)} - I \quad (33)$$

DE LA MARE has solved (33) numerically and demonstrated the relationship between P_0 and t_r for varying rates b of technological progress and discount rates r of return on investment when a and c are given. The values of r have been chosen so that (33) equates to zero. Not surprisingly, price P decreases with time due to technological progress. However, the initial price P_0 increases as the rate b of technological progress increases. At the same time, as an increased rate b of technological progress hastens the economic obsolescence, one has to pay for a shorter lifespan. A higher discount rate r will also cause a higher initial price P_0 but the economic lifespan will become longer.

In conclusion, the size parameter of the core WEIBULL distribution of lifetimes is, economically speaking, much more relevant than the shape parameter. It is indeed a crucial value which largely determines the amount to be depreciated once I is given.

V.5. Comparison with other Depreciation Models

Many depreciation (capital consumption) models have been devised for the estimation of the value of the stock of fixed capital assets. Many are concerned with accounting practice and are firmly based on company income and fiscal parameters. Generally, the latter category of models is not relevant to our study. Nevertheless, tax and subsidy policies may have a significant impact on capital consumption due to investment incentives as described by SCHWORM (1979) and others. Investment incentives are at the same time attacks on capital assets and durable products in service, resulting in an acceleration of the discarding process. This sort of attack is one of many types of shock of a stochastic nature which make lifetimes uncertain.

In Section III.2. a number of probabilistic lifetime distribution models were briefly discussed. One of them was developed by the U.S. DEPARTMENT OF LABOR & BUREAU OF LABOR STATISTICS (1979). Apart from the application of a two-sided vertically and of a horizontally truncated NORMAL distribution, this model find its rationale in a depreciation function which is of interest here. This is further elaborated in Section V.5.1..

Another interesting model deals with deterioration of roads, in particular, with pavement maintenance management with emphasis on planning and cost as described by KONING & MOLENAAR (1987). They measured the deterioration in the technical condition of several types of roads and pavements under different traffic loads and various local conditions. Their maintenance cost and planning model is analogous to a depreciation model because in this case technical condition losses over time are equivalent to losses in value over time due to use and continuous development of higher quality standards for pavement construction. The losses in value are reflected in the maintenance cost. Maintenance in this context also includes all the activities and physical flows required to maintain the function of a road in the most economic manner. As in the case of productive capital assets, maintaining the function means more than repair and restoration. Furthermore it is noted that roads represent capital providing a benefit flow over time: they are reproducible and depreciable. Roads are mostly financed and accounted for as current expenditure, which is more often than not government practice in the case of public assets. The road and pavement maintenance model is discussed in greater detail in Section V.5.2..

V.5.1. Depreciation Model of the US DEPARTMENT OF LABOR & BUREAU OF LABOR STATISTICS

The probabilistic scope of this depreciation model was discussed in Section III.2. above. In the description of this model in BULLETIN 2034 (1979) it is said that there is no general consensus as to what pattern depreciation follows with time. Two classes of depreciation patterns are considered. The first class comprises accelerated forms of depreciation where most of the performance decline occurs early in the service life of the productive capital asset. These forms, such as geometric decay, declining balance, sum-of-the-years digits, etc., are closely related to tax depreciation guide-lines and accounting principles. The second class of depreciation functions assumes that most of the depreciation occurs in the later years of service rather than in the early years. Given that both early and later depreciation could occur, a general depreciation function was developed which encompasses many different shapes of the depreciation pattern.

It is stressed that the development of a depreciation function was not based on the underlying hazard process of performance disruptions but on modelling a flexible depreciation form, so that by varying certain parameters, the depreciation pattern could be varied. This function is:

$$\frac{A - a}{A - B.a} \text{ (in the original notation)} \tag{34}$$

where:

A = mean service life of the capital asset (in years)

a = actual age for $a < A$

B = curvature parameter describing the form of depreciation for $B < 1$ (also negative for accelerated forms of early depreciation).

In fact (34) represents a ratio referred to in the BULLETIN 2034 concerned as an "efficiency under various depreciation assumptions". The "efficiency" referred to here is 1 at the age $a = 0$, and 0 at $a = A$. If $B = 0$, (34) corresponds to a straight-line or a linearly declining "efficiency" with age. The mean service life A is regarded as the mean of a horizontally or vertically truncated NORMAL distribution. A declining "efficiency" implies that the cumulative depreciation ratio increases. Then function (34) becomes:

$$\{[D/C](a|A,B)\} = 1 - \left(\frac{A - a}{A - B.a}\right) = \frac{a(1 - B)}{A - B.a} \tag{35}$$

Now it is postulated that the cumulative depreciation ratio function in our terms may be equivalent to the one developed by the US BUREAU OF LABOR STATISTICS, i.e., (1) and (35) are equivalent if:

$$\left(\frac{t}{\mu}\right)^{\beta} = \frac{a(1 - B)}{A - B.a} \qquad \text{for } 0 < a < A \quad \text{and } B \leq 1 \tag{36}$$

Substituting $a = t$ and $A = \bar{t} = \mu.\Gamma\{1 + (1/\beta)\}$, gives:

$$\left(\frac{t}{\mu}\right)^{\beta} = \frac{t(1 - B)}{\mu.\Gamma\{1 + (1/\beta)\} - B.t} \tag{37}$$

And for $t = \mu$, (37) becomes:

$$1 = \frac{1 - B}{\Gamma\{1 + (1/\beta)\} - B} \qquad \text{for } t = \mu \quad \text{and } B \leq 1 \tag{38}$$

Since $\Gamma\{1 + (1/\beta)\} = 1$ only holds if $\beta \rightarrow \infty$, equations (37) and (38) are inadequate. Therefore the substitution of A is replaced by $A = \mu$. Then (36) becomes:

$$\left(\frac{t}{\mu}\right)^{\beta} = \frac{t(1 - B)}{\mu - B.t} \qquad \text{for } 0 < t < \mu \quad \text{and } B \leq 1 \tag{39}$$

Equation (39) fits for $t = \mu$ and $t = 0$. Since $\bar{t} \approx 0.9\ \mu$, the cumulative depreciation ratio according to (39) is slightly higher than according to the original ratio as expressed by (35). This, however, is a desirable feature because the amount to be depreciated according to our capital consumption principles is higher than the initial (historical) investment used in the depreciation model developed by the US BUREAU OF LABOR STATISTICS. If (39) is taken as a point of departure, parameter B can be derived as follows:

$$B = \frac{\alpha^{\beta-1} - 1}{\alpha^{\beta} - 1} \qquad \text{for } 0 < \alpha < 1 \quad \text{and } \beta > 1 \tag{40}$$

where $\alpha = t/\mu$ and β is the shape parameter of a WEIBULL lifetime distribution. For $\beta = 1$, parameter $B = 0$, which is indeed equivalent to a linear cumulative depreciation ratio. Parameter B can be determined through a curve fit at the characteristic lifetime. Then, we have according to (III/18):

$$\alpha = \frac{t^*}{\mu_3} = \frac{\mu_3.\beta_3^{-1/\beta_3}}{\mu_3} = \beta_3^{-1/\beta_3} \qquad \text{,and thus:}$$

$$B = \frac{\beta_3^{(1/\beta_3)-1} - 1}{(1/\beta_3) - 1} \qquad \text{for } \beta_3 > 2 \tag{41}$$

In Table V-2 below parameter B is calculated for different shape parameters of the core WEIBULL lifetime distribution.

β_3	B
2	0.5858
3	0.7789
3.2589	0.8066
4	0.8619
5	0.9051
6	0.9304

Table V-2:
Value of parameter B for different shape parameters of the core WEIBULL distribution valid at the characteristic lifetime.

In BULLETIN 2034 (1979) of the US BUREAU OF LABOR STATISTICS the values chosen were B = 0.9 for structures and B = 0.75 for capital equipment. As can be determined by means of (41), these B-values correspond to $\beta_3 = 2.78$ and $\beta_3 = 4.90$ respectively which fall in the range found in this study.

Figure 16 on the next page shows the six pairs of cumulative depreciation (capital consumption) ratio curves referred to in Table V-2. Generally, the curves associated with our depreciation model (5) are more concave (when Phase II is disregarded) before the characteristic lifetime is attained and less concave thereafter than the curves associated with the adjusted US BUREAU OF LABOR STATISTICS model represented by (35).

The intersection of each pair of curves corresponds to the characteristic lifetime. At that moment the cumulative depreciation ratio amounts to $1/\beta_3$ as noted in Section V.3.1., formula (21). The shape of the curves reflects the cumulative depreciation (capital consumption) ratio pattern. The deviation within each pair is apparent but not great. Before the intersection the deviation rises to 6%; thereafter the deviation decreases monotonically to that level for $\beta_3 = 2$ to 4, and goes up to 15% for $\beta_3 = 6$ at $t = 0.9\ \mu_3$. It is stressed that the curves associated with the model developed by the US BUREAU OF LABOR STATISTICS are not based on a hazard process of performance disruptions but on modelling of depreciation assumptions. Nevertheless, they reflect the pattern of the ratio concerned fairly well.

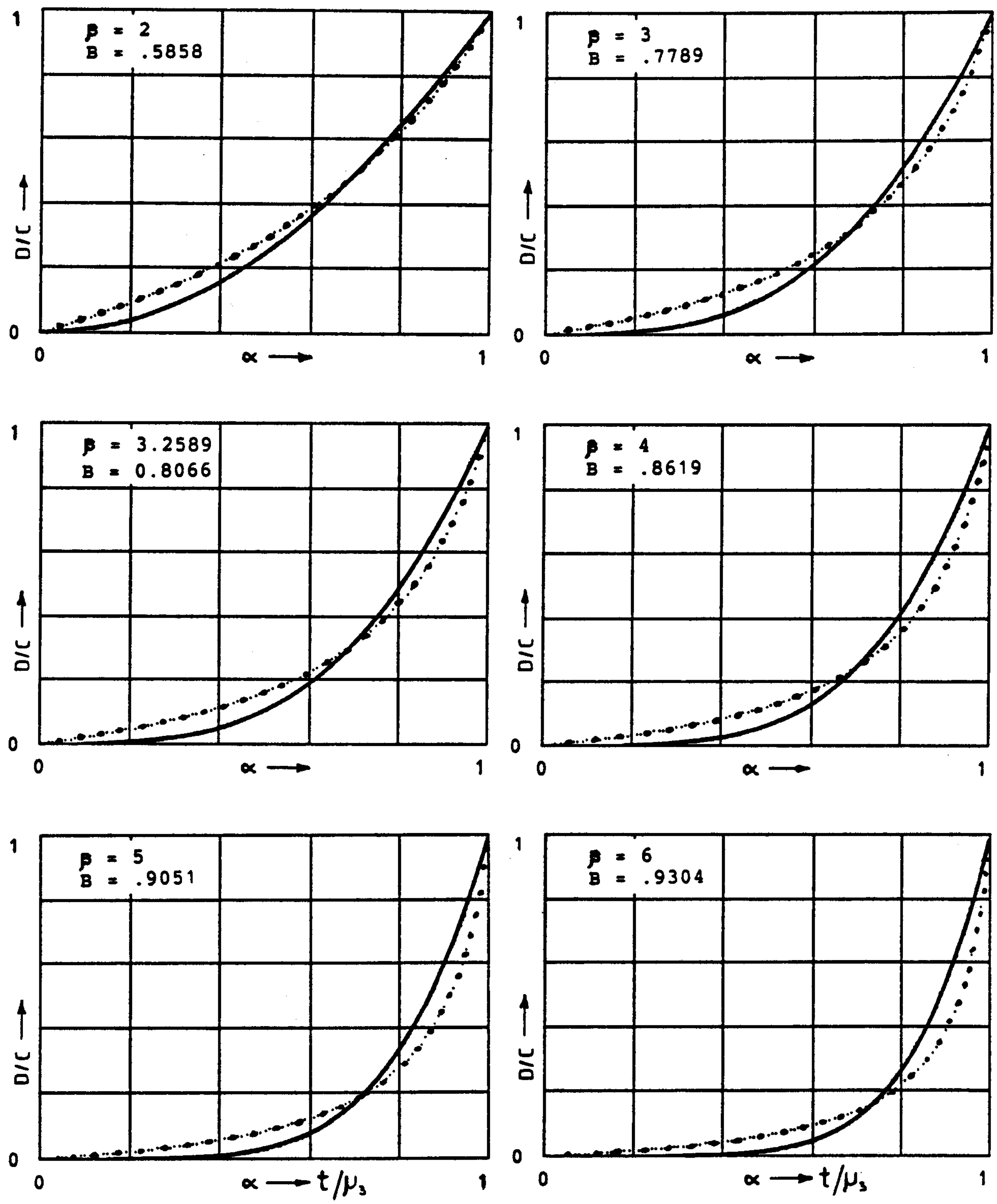


Fig. 16: Six pairs of cumulative depreciation ratio curves. The dotted curves are associated with the model of the US Bureau of Labor Statistics. The solid curves correspond to our basic model.

BULLETIN 2034 (1979) elaborates an example of an asset with a 10-year service life (mean). As a check, we used the cumulative frequencies obtained from that model and found an almost perfect WEIBULL distribution with shape parameter $\beta_3 = 5$ and size parameter $\mu_3 = 11.18$ years. The similarity of the two discarding and depreciation models associated with Phase III is striking in spite of the different points of departure.

V.5.2. Road Pavement Maintenance (Depreciation) Model

The deterioration of road pavement depends on a complex group of factors that are all related to ground and loading conditions, climate, construction and types of materials used. In our terms these damage factors are attacks of a stochastic nature by which the condition required to perform the function of a road and its pavement declines with time. This is not only valid for traffic lanes but also for parking lanes, bus-stops, bicycle lanes and pedestrian foot paths. The decline in condition is equivalent to loss in value, which can be measured by means of appropriate survey systems. All that is needed to maintain the condition for the required performance level can be regarded as depreciation, or in other words, as the capital consumption to maintain the road pavement stock intact and fit for its intended purpose.

KONING & MOLENAAR (1987) have analyzed condition-loss measurement data and models resulting in the following generalized mathematical concept:

$$P = 1 - (t/T)^a \quad \text{(in original notation)} \tag{42}$$

where:

P = cumulative loss of condition ratio

t = period between time of inspection and moment of last major maintenance work

T = period between last moment of major maintenance work and moment when P becomes zero

a = curvature parameter dependent on the type of defect

In this concept one recognizes immediately the similarity of:

$$(t/T)^a = (t/\mu)^\beta = H_W(t|\mu,\beta)$$

to the cumulative depreciation ratio in accordance with our concept.

On closer examination of the work of KONING & MOLENAAR (1987) it appears that they have disregarded Phase II, which is characterized by a linearly increasing loss of condition with time. Change failures caused by catastrophic events such as traffic accidents, severe weather conditions, unplanned excavations, etc., undoubtedly apply to road pavements, so that Phase II is applicable, but is not considered in their concept. The values they have found for the curvature parameter, a, coincide with our empirical and theoretical findings for shape parameter β_3 of the core WEIBULL lifetime distribution of capital assets. For instance, for cracking and ravelling defects in asphalt pavements these values are according to many empirical measurements:

$$a = 3.3 = \beta_3 \quad \text{and} \quad a = 3.5 = \beta_3 \text{ respectively.}$$

Furthermore, it is noted that cracking and ravelling defects can be justified theoretically on the principles of probabilistic fracture mechanics and crack propagation under cyclic loads. Comprehensive information on the subject of probabilistic mechanics can be derived from the relevant ASTM-Publication (1981).

PARIS & ERDOGAN (1963) have developed a relationship known in mechanical and civil engineering as the "PARIS LAW" crack growth rate equation:

$$dL/dN = E(\Delta R)^m \tag{43}$$

where:

L = characteristic crack dimension

N = number of stress cycles

E = constant (energy parameter) related to each particular (m)

ΔR = range of stress intensity factor R in consequence of stress variations

m = exponent related to E (constant within not too large a range)

If the stress intensity is taken to be proportional to $L^{\frac{1}{2}}$, crack growth rate equation (43) becomes according to NEWBY (1991):

$$dL/dN = \alpha . L^{\frac{1}{2}m} \tag{44}$$

where α is a constant. Integration of (44) gives the following crack length for $m > 2$ and $m = 2$ respectively:

$$L = \{L_0^{1-\frac{1}{2}m} - \alpha(\tfrac{1}{2}m - 1)(N - N_0)\}^{1/(1-\frac{1}{2}m)} \quad \text{for } m > 2 \qquad (45_1)$$

$$L = L_0.\exp[\alpha(N - N_0)] \quad \text{for } m = 2 \qquad (45_2)$$

where L_0 and N_0 are the initial values.

From (45) it can be calculated what the cumulative loss of condition (crack propagation) in the interval (N_0, N stress cycles) is if L_0, α and m are known. The quantity $(N - N_0)$ results from counting, probably a POISSON counting process of cyclic loadings or shocks, which is outside the scope of this study.

From the above it can be concluded that crack propagation induced by random shocks causes deterioration. The deterioration pattern as a function of N depends on α and m but the initial crack length, $L_0(N_0)$, equivalent to the initial condition, determines the origin of the deterioration (depreciation) curve. Consequently, the initial condition reflects the probability of survival at that point in the process and therefore determines the size parameter of the relevant lifetime distribution. This was also discussed at the end of Section III.6.4. with reference to the graphical presentation of the 3-component (composite) lifetime distribution model.

In this section it was demonstrated that our depreciation concept is analogous to technical and/or physical loss of condition which has economic consequences with respect to the loss of value of civil assets such as road pavements. Loss of value in this case is identical to the capital consumption associated with maintaining the function in the most economic manner. It was shown that the road pavement deterioration model developed empirically by KONING and MOLENAAR (1987) concurs with our theoretical approach. Consequently, for Phase III this model is identical to our concept.

Further confirmation is provided by DE KRAKER, TICHLER and VROUWENVELDER (1982) who have simulated lifetimes of structures on the basis of a crack growth law for creep. The result is indeed a perfect WEIBULL lifetime distribution. That case was also concerned with technical deterioration caused by stress and time dependent creep phenomena. If these destructions have economic consequences, deterioration is analogous to loss in economic value, which is equivalent to depreciation or capital consumption.

V.6. Amortization/Depreciation Model

The cumulative capital consumption associated with purchasing a given capital asset and with maintaining it in the condition required for competitive performance can also be regarded as a principal amount to be amortized. This means that fixed payments at regular intervals cover the interest on the principal outstanding and repay the principal over a given time span. The cumulated flow (interest + repayments) is equal to the capitalized value of the asset that produces the income stream needed to pay fixed periodical installments over a given number of years.

The leasing of capital equipment or, say, motor cars, or possibly, the output of a capital goods producing factory, etc., is founded on similar principles. The lessor provides the means to fulfil a production or service function in the most economic manner as specified by the user. The lease price of capital assets includes maintenance such that there is perfect substitution between new and older assets. This implies that competitive performance is ensured at every point in time. Discarding will take place only if upgrading to revised standards is not longer feasible. The above basis for leasing concurs with our point of departure. The lessor may be an external financial institution but could also be an internal financing division of a manufacturing or service company.

In any case a certain profit and return is required. In this context profits may be regarded as interest yielded from a principal which is the "loan" covering the purchase price as well as the cumulative amount required to maintain competitive performance at every point in time during the service life. The increasing proportion in the fixed periodic installments is regarded as depreciation or capital consumption. If so, then the "loan" stretches over a time span of a stochastic nature, because the service life is uncertain. The latter problem can be tackled by introducing our probabilistic lifetime distribution model. Consequently, the pattern of the remaining balance ratio may correspond to the pattern of the "net value ratio" of the (leased) capital asset as defined in Section V.3. (formula 12) above. The extent to which this assumption is correct is elaborated below by means of a mathematical amortization/depreciation concept.

V.6.1. Mathematical Concept

The remaining balance of the amount to be depreciated can be expressed in the formula for an amortization schedule:

$$\{[BAL](t)\} = (1 + i)^{t} . \left(PMT \frac{(1 + i)^{-t} - 1}{i} + PV\right) \quad \text{for } 0 < t < \mu \qquad (46)$$

where:

BAL = remaining balance or net value to both borrower and lessor

PMT = fixed periodic (financial) stream from the borrower to the lessor where a period is equal to one unit of time t

PV = present value of the future PMT-stream

i = periodic interest rate (fraction) desired for investment in capital assets and required to maintain competitive performance.

The fixed periodic amount is given by:

$$PMT = PV \left(\frac{i}{1 - (1 + i)^{-\mu}}\right) = PV.i. \left(\frac{(1 + i)^{\mu}}{(1 + i)^{\mu} - 1}\right) \qquad (47)$$

where μ is the number of periodic installments needed to achieve a zero balance equal to a zero net value at $t = \mu$. We choose μ to be the size parameter of the core WEIBULL lifetime distribution. At that point the cumulative depreciation ratio is 1. By combining formulae (46) and (47), and substituting $\mu = \mu_3$, we obtain the following remaining balance ratio:

$$\frac{BAL}{PV} (t|\mu_3,i) = \frac{(1 + i)^{\mu_3} - (1 + i)^{t}}{(1 + i)^{\mu_3} - 1} \quad \text{for } 0 < t < \mu_3 \qquad (48)$$

The factor (1 + i) may be regarded as a measure of productivity during one unit of time. In that time interval the input is one amount of capital and the output (1 + i) amounts. Capital is consumed to maintain this sort of productivity at a sufficient and constant level (1 + i). The associated capital consumption (depreciation) ratio is equal to one minus the "net value ratio" or "remaining balance ratio" represented by formula (48). Thus we obtain:

$$1 - \left(\frac{BAL}{PV} (t|\mu_3,i)\right) = \frac{(1 + i)^{t} - 1}{(1 + i)^{\mu_3} - 1} \quad \text{for } 0 < t < \mu_3 \qquad (49)$$

From (49) it follows that the ratio is zero when $t = 0$, and 1 when $t = \mu_3$. The pattern of the "capital consumption ratio" (cumulative depreciation ratio) is governed by the amortization parameters (1 + i) and μ_3. Now we can check how far (49) corresponds to our basic cumulative depreciation formula (5), by evaluating the following set of ratio functions:

$$\frac{(1+i)^{t}-1}{(1+i)^{\mu_3}-1} \div \left(\frac{t}{\mu_3}\right)^{\beta_3} \qquad \text{for } 0 < t < \mu_3,\ \beta_3 > 1 \text{ and } (1+i) > 1$$

When $t = \mu_3$, both quantities are equal and independent of the values of β_3 and i. After substitution of $t/\mu_3 = \alpha$ the above becomes:

$$\frac{(1+i)^{\mu_3 . \alpha}-1}{(1+i)^{\mu_3}-1} \div \alpha^{\beta_3} \qquad \text{for } 0 < \alpha < 1 \tag{50}$$

If the size and the shape parameter of the WEIBULL core distribution are given, parameter (1 + i) can be estimated. Conversely, if a certain (1 + i) is required, the size parameter can be estimated for a given shape parameter. To evaluate the parameter, equation (50) can be used, after substituting:

$\alpha^{*} = \beta_3^{-1/\beta_3}$, valid at the characteristic lifetime $t = t^*$.

Then (50) becomes:

$$\frac{(1+i)^{\mu_3 . \beta_3^{-1/\beta_3}}-1}{(1+i)^{\mu_3}-1} = 1/\beta_3 = H_W(t^*) \tag{51}$$

Equation (51) represents the relationship between the three parameters (1 + i), μ_3 and β_3. From (51) the amortization parameter (1 + i) is calculated for: μ_3 = 20 years and β_3 = 2, 3, 3.2589, 4, 5 and 6. The results are summarized in Table V-3 below.

Size Parameter μ_3 = 20 years	
β_3	(1 + i)
2	1.10
3	1.19
3.2589	1.20
4	1.26
5	1.34
6	1.41

Table V-3:
Values of amortization parameter (1 + i) for different shape parameters of the core WEIBULL distribution with size parameter of 20 years.

The resulting cumulative depreciation (capital consumption) ratio curves are illustrated in Figure 17 on the next page.

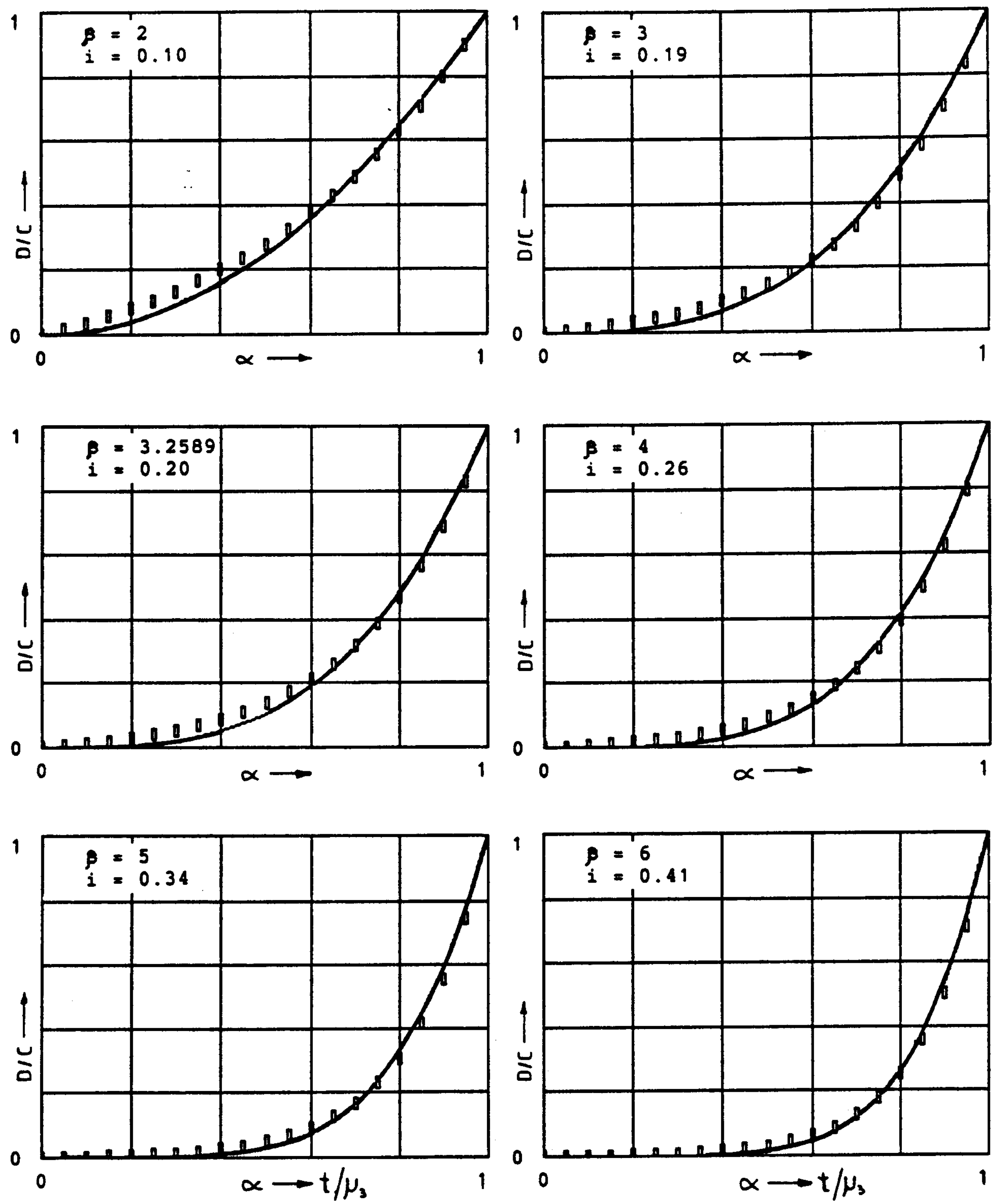


Fig. 17: Six pairs of cumulative depreciation ratio curves. The dotted curves are associated with the amortization model. The solid curves correspond to our basic model.

Each plot contains a pair of curves. The solid curve represents the ratio related to Phase III of our model and the dotted curve represents the amortization model. All the plots demonstrate a good fit for each of the pairs of curves. The fit looks even better than those in Figure 16 associated with the US BUREAU OF LABOR STATISTICS. Since both ratio functions represented by (50) are related to productivity, the better fit is not surprising.

Although $(1 + i)$ may be regarded as a capital productivity measure per unit of time in time interval $(0, \mu_3)$, it is premature to conclude that a high valued shape parameter is always favourable. For example, a value of $(1 + i) = 1.26$ holds for $\beta_3 = 4$ and $\mu_3 = 20$ years, but according to (51) also for $\beta_3 = 2$ and $\mu_3 = 8.2$ years. Furthermore, it is stressed that the shape parameter tends to a value that deviates not much from $\beta_3 \approx 3.26$. A lower value is more often than not associated with an aggregated mass and/or inaccurate lifetime measurements.

We note that the amortization model discussed above is a single-life-phase concept which offers no provision for Phases I and II. It is closely related to the (lease) price of capital services. In this respect the amortization model coincides with the view of JORGENSON (1974) who defines depreciation as utilization of capital, which he equates with the "rental price of capital services". It is also related to the replacement model developed by MALCOMSON (1975) who defined the rental value as the current cost attributed to the use of capital equipment.

In this section it is demonstrated that the amortization/depreciation/ capital consumption model developed, is in practical terms, similar to our basic model in relation to Phase III. There is a robust parametric relationship between the size and the shape parameters of the core lifetime distributions on the one hand and the profitability (productivity) of capital services on the other.

CHAPTER VI

SUPPLEMENT, SUMMARY AND FINDINGS

VI.1. Interpretation of some Economic Replacement Models

In the previous chapters the lifetime of capital assets was considered from a probabilistic point of view, which led to the development of our 3-component (composite) WEIBULL model of lifetimes and to an associated depreciation (capital consumption) model. In our concept the question of whether or not a depreciable and reproducible capital asset or durable (industrial) product should be replaced at the end of its service life is not relevant. As stated earlier in this study, there are just two options: maintaining or discarding from the corresponding class of stock. In our terms the performance rate is the decisive factor in the decision to maintain the asset or product in question or to replace it. There is continuous deliberation as to which of the options is preferable. Consequently, the timing of replacement may depend on (dynamic) performance criteria. If the timing of replacement takes place solely on the basis of economic criteria, the optimal service duration may be obtained by means of deterministic/vintage replacement estimation models. These models are based on economic factors which are determinants in the maintenace/replacement decision-making process. Deterministic/vintage replacement models more often than not contain all kind of assumptions (including probabilistic ones) for which arbitrary provision must be made. Of course, the implications of such assumptions can be subjected to sensitivity and probability analyses, but it cannot be denied that many deterministic/vintage models are affected by a greater or lesser degree of uncertainty. This is not, or not always, a serious problem as long as the degree of uncertainty is known. In this section some replacement models will be interpreted in the light of our concept developed above.

Up to 1970 it was widely assumed that replacement of capital investment was proportional to the capital stock, and moreover, that the average replacement rate, in the long run, was constant. JORGENSON (1965) demonstrated this on the basis of renewal theory on the assumption that the capital stock is growing at a constant rate. FELDSTEIN and FOOT (1971) empirically demonstrated that the replacement rate concerned varied from year to year between 0.0499 and 0.0718 over a period of nineteen years (1949-1968). They concluded that replacement investment varies around some non-zero level in a way which is systematically related to other short-term economic forces. According to FELDSTEIN and FOOT the exact timing of replacement is substantially influenced by three factors that vary cyclically: the availability of funds, the desire for expansion investment, and short-term pressure to reduce production costs. Potential opportunities play an essential role in replacement investments and, consequently, plant and production equipment are replaced when the balance of economic forces makes that decision most profitable.

FELDSTEIN and ROTHSCHILD (1974) presented evidence that the replacement ratio is not constant as supposed by JORGENSON (1965). This ratio remains constant only if the output decay of each vintage occurs at some constant rate and, in addition, the age composition of the capital stock remains unchanged. Furthermore the latter condition also requires an exponential growth rate. The exponential rates mentioned above are not only contrary to the facts, they also obviate the need for a replacement theory. Because if the service life of capital assets were characterized by an EXPONENTIAL distribution, there would be no optimal replacement age. FELDSTEIN and ROTHSCHILD also refer to WINFREY's survivor curves which are all non-exponential. We note that our probabilistic lifetime model provides for an EXPONENTIAL distribution during Phase II when replacement is the consequence of a time-independent hazard rate which lead to total loss of performance. Phase II is relatively significant for motor vehicles, as we saw in Sections IV.6., IV.7. and IV.8. which may be why the exponential decay assumption originates from studies in that field. FELDSTEIN and ROTHSCHILD referred to the class of capital assets in which all output decay occurs at one time, i.e., "one-horse shay" deterioration. They maintain that such assets deliver a constant amount of output over their service life and generate no replacement until they fail suddenly. Such assets, defined as "failing goods", have no output decay because they do not wear out during service and, as a consequence, the scrapping age is time-independent. However, this is not true because non-repairable or non-maintainable systems do not fail solely due to exponentially distributed events. Electric and telephone cables or electronic systems do not technically wear out and so belong to the category of non-repairable and non-maintainable failing goods, but the scrapping age is not time-independent because of changing economic, technological, social and environmental parameters. Even new electric telephone cables are replaced by advanced glass-fibre cables in the wake of technological progress. The same applies to electronic equipment like computers and measurement and control systems. A simple light bulb will only not be replaced when its filament burns out, but also when more efficient illumination systems become available. As demonstrated in Chapter IV, the lifetime of failing goods is also a stochastic variable, and in Phase III it is characterized by an increasing hazard rate. FELDSTEIN and ROTHSCHILD have studied replacement ratios assuming two stochastic decay functions: the PASCAL distribution and the EXPONENTIAL distribution. In simulations using the two-parameter PASCAL distribution, they assumed that the date of final discarding was sufficiently distant to ensure that almost all of the capital assets would have decayed before that time. The results of 7 simulations (almost EXPONENTIAL, 3 runs with truncated EXPONENTIAL with different parameters, 3 runs with truncated higher-order PASCAL) selected from a much larger set reflect the firm assumptions that replacement is equivalent to output decay. The coefficients of variation rise to between 5 and 14.6 percent for the truncated EXPONENTIAL and to between 6.1 and 12.3 percent for the truncated PASCAL distribution. The departures from zero imply rejection of a constant replacement ratio. Although these departures are relatively small, they result in substantial fluctuations in replacement investment. They maintained that the need for adequate survivor functions was apparent. The results of our study may bridge that gap.

In Section V.6.1. we referred to JORGENSON (1974) who published an eccnomic theory of replacement and depreciation. On the basis of the renewal theory he demonstrated that the "mortality distribution" is geometric and the average replacement rate is constant. A fundamental characteristic of renewal theory is that the sequence of replacement rates tends to a constant value for almost any mortality distribution. According to JORGENSON, the usefulness of the geometric approximation depends on the speed of convergence of the replacement distribution to its constant asymptotic value and on the variation in the weights that determine the average replacement rate. We note that the short-term dynamics of replacements are a serious source of disruption in JORGENSON's economic theory of replacement and depreciation.

The timing of replacement depends on several dynamic factors such as adjustment costs, financing preferences, internal available funds and fluctuations in expansion investment. NICKELL (1975) demonstrates that a reduction in user cost can/will lead to increasing internal funds and, consequently, to a rise in replacement and expansion investment. Typically, replacement will be high both before and after a boom in product demand and low during the boom itself. In an expansionary period the existing capacity is likely to be stretched before the new capacity is provided. Replacement investment and expansion investment are inversely related if user cost is kept constant. As well as the bunching effects of changing demand, we must also consider the influence of technical change on the dynamics of replacements. As we argued in Section V.3.1., technological progress is a time-dynamic mechanism that can be regarded as a stochastic point process. Consequently, the state-of-the-art in technology has a lifetime which is a random variable. However, the dynamics of replacement are beyond doubt.

NICKELL (1975) developed a comprehensive vintage replacement model based on the concept of maximizing the present value at every point in time. According to NICKELL a unit of capital is discarded from stock at the point in time when its quasi-rent is zero. Then, marginal revenue is equal to marginal maintenance cost plus marginal wage cost. That constitutes his "scrapping rule" wherein a maintenance function with time is assumed. That maintenance function covers any costs which are incurred to maintain the initial productivity. Technological progress is covered by a factor which represents the elasticity of output/capital ratio. He demonstrates that, in economic terms, the lifetime is only constant if factor prices, the discount rate and the elasticity of the output/capital ratio are all constant. The output/input ratio is identical to the constant rate of neutral technological progress as developed by HICKS (1965). Under the condition that the labour/capital ratio is one, the wage rate is constant, the rate of interest is constant, the elasticity of the output/capital ratio is constant and equal to the constant rate of embodied HICKS neutral technological progress, and assuming that the maintenance cost increases exponentially with time, NICKELL derived the following simple approximation for the optimal lifetime:

$$m^* \approx \left(\frac{2.v}{(\eta + \omega)\phi_0.c + \eta.w)} \right)^{\frac{1}{2}} \qquad (1)$$

where (in original notations):

m^* = optimal lifetime in years

v = price of the capital asset in money terms

w = wage rate in costs per year

η = constant rate of embodied HICKS neutral technological progress per year

ω = rate of increase of maintenance per year

$\phi_0.c$ = maintenance element in costs per year

In fact, formula (1) is typical of other results derived from optimal durability theory and supports NICKELL's results.

NICKELL investigated the implications of a constant replacement/capital ratio, as assumed by JORGENSON (1974) and others. He demonstrated that both the elasticity of demand function and the growth rate of demand must be constant for the ratio of replacement to capital stock to be a constant. This implies that the rate of investment increases evenly. Using his notation, he found that:

$$r = \frac{\eta}{\exp[\pi.m^*] - 1} \tag{2}$$

where:

r = replacement/capital ratio

π = investment growth rate

As we can see, the rate decreases in π, increases in η and is only constant if both $\pi.m^*$ and η are constant, which is only likely to be true on some long-term average basis. A faster growth rate in output and a higher elasticity of demand both imply a smaller ratio of replacement investment to capital stock in the steady state. The exponential decay assumption, as applied by JORGENSON (1974) and others, is clearly inadequate.

MALCOMSON (1975) also developed a vintage model to support an optimum replacement policy. The model aims at maximizing the present value of the net revenue stream over an infinite horizon from a starting date. Equipment of a given vintage should be used only as long as the operating cost of producing a unit of output on equipment of that vintage is less than the marginal cost of producing that output on the most recent vintage. This discarding rule or equation involves the age of the oldest equipment and the optimal lifetime of current equipment. It is impossible to determine the optimal lifetime of current vintages of capital equipment without knowing the optimal service life of those that will replace them. MALCOMSON solves that problem by means of an iterative process based on restricting the optimal life and on a construction to derive admissible values concerning the discounted value of the marginal costs of the output (at a given time) per unit of output. The latter quantity comprises also the implicit rental value which is the current cost attribution to the use of capital equipment. This rental value may correspond to what we have discussed in Section V.6.1. dealing with an amortization/depreciation model.

MALCOMSON demonstrated that a constant optimal life of capital assets as derived by SMITH (1961), can be valid only under the condition that (in original notations):

$$dT(t)/dt = dL(t)/dt = 0$$

where:

$T(t)$ = age of the oldest vintage in use at point t in time

$L(t)$ = optimal lifetime = $T\{t + L(t)\}$; thus $T(t) = L\{t - T(t)\}$

When SMITH's assumptions are taken into account, $T(t)$ becomes according to MALCOMSON:

$$T(t) \approx [2q/(\alpha + \beta)]^{\frac{1}{2}} = P \tag{3}$$

which is the formula derived by TERBORGH (1949/1958) for the pay-off period P,

where:

α = measure of technological progress in costs per unit of output per square year

β = measure of deterioration in costs per unit of output per square year

q = prices of capital equipment per unit of output (assumed to be constant).

Hence, $T(t)$ as defined by MALCOMSON is an approximation of TERBORGH's pay-off period P when SMITH's assumptions are taken into account. The standard form of pay-off period criteria is that equipment should be made to pay for itself over a certain period P, not necessarily equal to its actual lifetime. MALCOMSON (1975) demonstrated that, using SMITH's assumptions, the discarding age becomes:

$$T(t) = (1/P[q/(\alpha + \beta)] + \tfrac{1}{2}P \tag{4}$$

Since the right hand side of (4) is independent of time, $T(t)$ will be constant over time. The exact value of $T(t)$ now depends on the pay-off period, P, chosen. One standard procedure is to let the pay-off period be equal to the reciprocal of the discount rate. See SMITH (1961); pp. 224-228. This allows the discount rate r to be brought in by substitution of $P = 1/r$. Then it follows that:

$$T(t) = [q.r/(\alpha + \beta)] + (1/2r) \qquad \text{for } P = 1/r \tag{5}$$

Using SMITH's assumption and the results obtained above in MALCOMSON's model, the following age for discarding capital equipment is obtained:

$$T(t) = \frac{1}{(\alpha + \beta)} \{r.q + \{1 - \exp[-r.L(t)]\}\frac{(\alpha + \beta)}{r}\}$$

$$= \frac{r.q}{(\alpha + \beta)} + (1/r)\{1 - \exp[-r.L(t)]\} \tag{6}$$

Combining (5) and (6) gives:

$$1 - \exp[-r.L(t)] = 0.5, \text{ and thus:}$$

$$L(t) = -(1/r).\ln 0.5 = -P.\ln 0.5 = 0.693\ P \tag{7}$$

Given the assumptions, MALCOMSON's conclusion as expressed by formla (7) is equivalent to our conclusion as elaborated in Chapters II and V:

- $L(t)$ amounts to approximately 70% of the age, $T(t)$, of the oldest capital equipment installed, or in TERBORGH's terms, 70% of the pay-off period P.

If TERBORGH's pay-off period P is interpreted as the moment at which the cumulative depreciation (capital consumption) ratio $D(t)/C(t) = 1$ (see Section V.3.1., formula V/24 and Figure 15), we obtain the equivalence:

$$T(t) = P = \mu_3 \tag{8}$$

which is the size parameter of the core WEIBULL distribution in our model. Furthermore, this is the point in time when the probability of an innovative occurrence attains its maximum (Section V.3.1.). The relationship between MALCOMSON's vintage model (via SMITH's assumptions) and our probabilistic model is a significant finding.

We referred above to TERBORGH (1949/1958) who introduced a replacement criterion called "adverse minimum" which can be regarded as a measure of capital performance over the lifetime of investments in productive assets. Although TERBORGH's replacement criterion is not relevant for our purpose, we may refer to his idea of "operating inferiority" which is represented as an increasing function of time t, and equal to zero when $t = 0$. Operating inferiority is a consequence of deterioration and obsolescence. TERBORGH defined deterioration as a decreasing performance with time if compared with identical new assets. Our concept of performance as defined in Section II.2., differs because we compare the productivity of an existing capital asset with the productivity of the newest capital asset (Challenger) embodying the newest state-of-the-art in technology.

ARROW (1962) has developed a model which is based on the principles of the learning curve (L.C.) as empirically derived by WRIGHT (1936). The L.C. is discussed in Section VI.2.1.2.. ARROW assumed that a new capital good will always be used in preference to an older one because of its higher productivity.

Learning and experience are always incorporated into the newest capital equipment, which reflects technological progress. ARROW's model ignores the capital-labour substitution, because according to him there is only one efficient capital/labour ratio which is open to the entrepreneur's choice at the time of investment, but is fixed once the investment is congealed into a capital asset. Hence, the stream of potential profits depends merely upon expectations of future wages. He stated that entrepreneurs assume exponentially rising wages from the level at the moment a new unit of capital is installed. The profit at time t from a unit of investment made at time $v \leq t$, according to ARROW, is (in original notation):

$$\text{Profit}(t) = \Phi[G(v)] - w(t).\lambda[G(v)] \tag{9}$$

where:

$\Phi[G(v)]$ = output capacity of a capital asset of serial number (G) which is a constant

$w(t)$ = wage rate at time t related to the output as numéraire

$\lambda[G(v)]$ = amount of labour used in production with a capital asset of serial number G which is a decreasing function of G of the form associated with the L.C..

Looking ahead at any given moment of time it is assumed that wages will rise exponentially from the present level. Thus the wage rate expected at time v to prevail at time t is:

$$w(t) = w(v).\exp[\theta(t - v)] \tag{10}$$

where θ is the wage growth rate.

In accordance with the L.C. function the labour costs over output at time v are:

$$W(v) = \frac{\lambda[G(v)].w(t)}{\Phi[G(v)]} \tag{11}$$

If w(v) is replaced by (10), we obtain the labour cost over output at time t:

$$W(t) = \frac{\lambda[G(v)].w(v).\exp[\theta(t - v)]}{\Phi[G(v)]} \tag{12}$$

Then the profit expected at time v to be received at time t can be obtained by combining (9) to (12):

$$\text{Profit}(v) = \Phi[G(v)]\{1 - w(v).\exp[\theta(t - v)]\} \tag{13}$$

From the above profit function it can be seen that the profitability of an investment is expected to decrease with time if $\theta > 0$ and to reach zero at the economic lifetime. ARROW defined this by the following equation:

$$W.\exp[\theta.T^*] = 1 \tag{14}$$

where T* is the expected economic lifetime, provided it does not exceed the physical lifetime. In other words, T* represents the expected date of obsolescence. From formula (14) it follows that:

$$T^* = -(1/\theta).\ln W \tag{15}$$

Formula (15) is indentical to formula (7) of MALCOMSON (1975), if SMITH's assumptions are taken into consideration where:

- the expected economic lifetime is constant, as assumed by both ARROW and SMITH (1961), and
- the wage growth rate θ is equal to:

$$\theta = \ln W/P.\ln 0.5 = r.\ln W/\ln 0.5 \tag{16}$$

From ARROW's formula (15) we can conclude that the expected economic lifetime increases when the labour cost growth rate decreases and when the labour cost fraction of the output decreases. This conclusion is in agreement with MALCOMSON's findings discussed above.

The discounted stream of profits S over the effective lifetime T of an investment, according to ARROW's model is:

$$S = \int_0^T (\exp[-r.t].\Phi[G(v)]\{1 - W.\exp[\theta.t]\})dt \tag{17}$$

Then, the discounted stream of profits over output capacity becomes:

$$\frac{S}{\Phi[G(v)]} = \frac{1 - \exp[-\theta.T]}{r} + \frac{W\{1 - \exp[-(r - \theta)T]}{\theta - r} \tag{18}$$

The dynamics of replacement are demonstrated by equation (18) that represents a ratio of the discounted stream of profits to the output capacity. According to (18), the effective lifetime T increases when the discount rate r is higher, the wage rate W and the wage growth rate θ are lower, and conversely. The wage growth rate may be in line with the rate of technological progress as the ratio between the labour costs and the labour productivity of the latest capital equipment remains unchanged. Both aspects are inherent to the principles of the L.C.. The wage rate W refers to the starting point of the L.C. related to that particular kind of capital asset. When the wage rate W in the ouput is lower, the capital coefficient will be higher, and conversely. Consequently, we may conclude that the effective lifetime T increases when the discount rate and the capital coefficient are higher, and the rate of technological progress is lower, and vice versa.

The influence which technological progress and the cost of capital have on replacement is evident. Since the latter factors are of a dynamic (and stochastic) nature, the replacement model according to (18) is a narrow tool. For that reason, VAN HULST (1973) and also DE LA MARE (1982) introduced a dynamic programming approach to replacement analysis which is, however, out of the confines of our thesis.

MALCOMSON's model is probably one of the most appropriate vintage models to determine the optimal lifetime of capital assets on the basis of purely economic factors. Meanwhile we know that many other factors of a stochastic nature can have a considerable impact on life characteristics. Therefore a probabilistic model may offer a more adequate tool to deal with the uncertainty attendant on the service lives of capital assets. The common shortcoming in the vintage and deterministic models discussed above is that they do not provide for successive and distinctive life phases with differently valued parameters.

VI.2. Further Research

In this study the importance and usefulness of an appropriate lifetime distribution model for depreciable and reproducible capital assets and manufactured durables became obvious. When our model is taken as a point of departure, it may be relatively easy to set up a lifetime database which contains relevant information to confirm and to improve our model. As a minimum requirement, it seems sufficient to know the size parameter μ_3 of the core WEIBULL distribution, i.e., the age when the probability of survival amounts to 0.368; in other words, when the ratio depreciation over cumulative capital consumption [D/C] = 1. Since $\bar{t} \approx 0.9\ \mu_3$, the average lifetime datum $\bar{t}$ is also sufficient in completing an approximation to our model. Because on the basis of:

$$\mu_3, \text{and } \beta_3^*[3] = 3.2589 \text{ (most likely value)},$$

the following parameters can be estimated:

$$\mu_2 \approx \mu_3^2 \quad \text{and } a = \mu_3^{(\beta_3-2)/(\beta_3-1)} \approx \mu_3^{0.56}$$

and the following quantities can be estimated:

- characteristic lifetime : $t^* \approx 0.7\ \mu_3$
- hazard rate at $t = t^*$: $h(t^*) = 1/t^* \approx 1.43/\ \mu_3$
- integrated hazard at $t = t^*$: $H(t^*) = 1/\beta_3 \approx 0.307$
- integrated hazard at $t = 1$: $H(1) = 1/\mu_3^2$
- probability of survival at :
  - $t = 1$: $S(1) = \exp[-H(1)] = \exp[-(1/\mu_3^2)$
  - $t = a$: $S(a) = \exp[-(a/\mu_3^2)] \approx \exp[-\mu_3^{-0.44}]$

Of course, the shape parameter of the core WEIBULL distribution can be classified within the range $2 < \beta_3 < 6$ according to the class of capital assets and taking into account aggregation and measurement errors. The above would be an interesting subject for further research.

Since the size parameter mentioned above plays a crucial role in our model, further research is recommended that lays emphasis on factors or functions which determine its value. Then it may be possible to classify the size parameters as well.

The application of our model in deriving a universal depreciation methodology (Chapter V) gives also rise to further research. One of the questions is how universal it is or, perhaps, how specific and restrictive.

One of the most essential findings of this study is concerned with the integrated hazard at the characteristic lifetime $H(t^*)$ when $h(t^*) = 1/t^*$. In the next two subsections an attempt is made to interprete $H(t^*)$ as a capital elasticity. For that purpose in Section VI.2.1.1. the COBB-DOUGLAS function is employed. In Section VI.2.1.2. the empirical learning curve equation is considered in that respect.

VI.2.1. Towards Capital Elasticity

In Section V.3. we derived the following form of the "net value ratio":

$$\{1 - \ln[C(t)/I]\} = H_w(t^*).\pi_w(t) \qquad (V/13)$$

where $H_w(t^*) = 1/\beta_3$ which is the depreciation ratio at the point in time when the average total capital consumption (per unit of time) is reduced to a minimum. From (V/13) it follows that:

$$H_w(t^*) = \frac{\{1 - \ln[C(t)/I]\}}{\pi_w(t)} = \frac{1 - H_w(t)}{\pi_w(t)} \qquad (19)$$

which is the WEIBULL integrated hazard at $t = t^*$ that reflects the rate of change with time t of two quantities. This is characteristic for the definition of an elasticity. Therefore, we may term $H(t^*)$ as a capital elasticity. If so, then $H(t^*)$ may be employed in the COBB-DOUGLAS production function and in the equation of the learning curve (L.C.) as developed by WRIGHT (1936). This is elaborated in the next subsections.

VI.2.1.1. Relation with COBB-DOUGLAS type Production Function

In Section II.2. and thereafter we argued that discarding due to economic obsolescence may be regarded as discarding due to productivity obsolescence. In both cases the conditions are maximization of profit by entrepreneurs and competition in product markets. This implies that the present value of production at every point in time is maximized. Then the ratio of the marginal labour productivity to the marginal capital productivity is equal to the ratio of the wage rate to the capital price:

$$(\partial Y/\partial L)/(\partial Y/\partial K) = w/\rho_c$$

where:

Y = output volume of capital assets

L = labour volume involved in production

w = wage rate involved in production

ρ_c = factor price of capital (c in volume)

The following simple production function of the COBB-DOUGLAS type is taken as a point of departure:

$$Y(t) = K_t^{\varepsilon}.L_t^{(1-\varepsilon)} \tag{20}$$

where:

$Y(t)$ = gross production output as a function of time (years)

K_t = capital invested (installed) in year t

L_t = labour volume involved in production in year t

ε = elasticity of capital for $0 < \varepsilon < 1$, and thus, $0 < (1-\varepsilon) < 1$ which is the elasticity of labour

From the COBB-DOUGLAS production function (20) it follows that:

$$\frac{\Delta Y_t}{Y_t} = \frac{\varepsilon.\Delta K_t}{K_t} + \frac{(1-\varepsilon).\Delta L_t}{L_t} \qquad \text{,and thus}$$

$$\left(\frac{\Delta Y_t}{Y_t} - \frac{\Delta L_t}{L_t}\right) = \varepsilon.\left(\frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t}\right) \tag{21}$$

After substituting:

$$\left(\frac{\Delta Y_t}{Y_t} - \frac{\Delta L_t}{L_t}\right) = \dot{Y}_t = \text{productivity growth with time t,}$$

and $$\left(\frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t}\right) = \dot{k}_t = \text{capital intensity growth with time t,}$$

equation (21) leads to the following productivity growth function:

$$\dot{y}_t = \varepsilon.\dot{k}_t \tag{22}$$

In this productivity growth function ε is an elasticity of capital which may be identical to $H(t^*)$. This seems to be reasonable because the underlying discarding process is mainly governed by technological progress leading to productivity growth. If so, then we have:

$$\varepsilon = H(t^*) = 1/\beta_3 \tag{23}$$

This result can now be introduced to the COBB-DOUGLAS family of production functions and associated models of the fundamental shape:

$$K^{(1/\beta_3)} . L^{1-(1/\beta_3)} \tag{24}$$

COBB and DOUGLAS (1928) estimated for the U.S.A. at that time a capital elasticity of 0.25 that, according to our findings, results in $\beta_3 = 4$.

For $\beta_3 = 2$ (limiting case in our model) the capital and labour elasticity are equal (0.5).

For other typical and characteristic β-values derived theoretically in Section III.5., the corresponding capital elasticities are:

$\beta_3 = 2 \rightarrow \varepsilon = 0.5$ (threshold value)

$\beta_3 = \exp[1] \rightarrow \varepsilon = 0.368$ (if $t^*/\mu_3 = H(t^*)^{H(t^*)}$ = minimum)

$\beta_3^*[1] = 3.05 \rightarrow \varepsilon = 0.328$

$\beta_3^*[2] = 3.259 \rightarrow \varepsilon = 0.307$

$\beta_3^*[3] = 3.667 \rightarrow \varepsilon = 0.273$

These values agree well with the capital elasticities found in macro-econometrics. Of course, this is not a formal proof of correctness but an interesting subject for further research. Assuming that (23) is correct, it can be derived from (20) that:

$$\varepsilon = \frac{\ln[Y(t)/L_t]}{\ln(K_t/L_t)} \quad \text{,and thus}$$

$$-H_w(t^*) = \frac{\ln[L_t/Y(t)]}{\ln(K_t/L_t)} \tag{25}$$

Since $S_w(t^*) = \exp[-H_w(t^*)]$, it follows from (25) that:

$$S_w(t^*) = [L_{t^*}/Y(t^*)]^{1/\ln(K_{t^*}/L_{t^*})} \tag{26}$$

Herewith the probability of survival of capital at $t = t^*$ is expressed in terms of a COBB-DOUGLAS type production function and amounts to approximately 0.7 to 0.8 for $\beta_3 \approx 2.8$ to $\beta_3 \approx 4.5$ respectively.

Since the learning curve (L.C.) characteristics are also based on productivity growth due to learning, experience and progress, the progress elasticity may be related to ε and thus to a COBB-DOUGLAS type production function (20). This subject is discussed in the next subsection.

VI.2.1.2. Learning/Experience/Progress Curve

The phenomenon that direct labour time involved in the mass-production of identical items reduces as experience in making it increases, was actually discovered in the twenties. Many of the techniques of mass-production were pioneered by FORD on the assembly line used to manufacture the famous Model-T automobile.

ABERNATHY & WAYNE (1974) analyzed the FORD experience on the basis of the well-known learning curve which was first reported in the literature by WRIGHT (1936). WRIGHT's learning curve is a plot of labour time per item against serial number; when plotted on a double log-grid for both the horizontal and vertical axis, the learning curve is a declining straight line. This concept has been comprehensively documented in the specialist literature. YELLE (1979) has made a historical review and comprehensive survey of the learning curve. The idea is simple: as workers learn an operation, their experience increases and their performance improves. As a result, the direct labour input per unit declines, which leads to manufacturing progress and a systematically cost reduction in mass-produced items.

According to the BOSTON CONSULTING GROUP, BCG,(1970) the implications go far beyond the mere prediction of labour costs. Their "experience curve" encompasses all manufacturing costs (including capital, administrative, research and marketing) and traces its effect through technological displacement and product evolution. It implies that the cost of doing any task of a repetitive nature, not necessarily production, decreases as accumulated experience in doing that task increases. BCG claims that the average costs (expressed in constant prices) of most value-added items decline consistently 20% to 30% each time the accumulated physical output volume is doubled. This concept is a simple but powerful strategic tool in forecasting future costs (in constant prices) of standard items. Improvements in labour efficiency, economies of scale and technological progress can be derived from or predicted by the experience curve as a sequence in the logical step-by-step development aimed at competitive performance. Another fundamental point worth noting is that the experience curve of a standard product is connected with the product's life cycle, as demonstrated by YELLE (1983). Here we meet the combined effect of all kinds of process and product innovations reflected by the empirical concept of the learning/experience/progress curve.

HIRSCH (1956) applied the learning curve to interpret "firm ratios" by introducing a progress function from which a progress elasticity can be derived. The percentage decline in the direct labour requirement associated with a doubling of the accumulated physical output (volume) was referred to as the "progress ratio". The relative rate of change was referred to as the "progress elasticity". He proved mathematically that the slope of the learning curve corresponds to the progress elasticity. NADLER & SMITH (1963) have extended the more theoretical work of HIRSCH and others. They tried to decompose the manual, mechanical and, in particular, process design elements of the learning curve. Evidence was found to substantiate the progress function effect in all the cases studied. It appeared that each basic operation used in the manufacture of a product has its own progress function. The integral progress function represents, generally, the combined effect of learning, experience and, consequently, all types of progress which are reflected in productivity and performance.

The functional relationship commonly specified for WRIGHT's empirical concept is represented by the following mathematical expression:

$$\bar{y} = A.x^{b} \quad \text{for } b < 0 \tag{27}$$

where:

$\bar{y}$ = average cost (in constant prices) or average labour input for any (subsequent) quantity x

A = inputs (factor costs or manhours) to manufacture the first standard unit or quantity

x = number of completed standard units or quantities = cumulative output

b = the slope of the learning curve (negative exponent)

From function (27) it follows that:

$$\ln \bar{y} = \ln A + b.\ln x \tag{28}$$

representing the L.C. as a straight line. Since b must be negative in order to obtain a positive effect of learning, this line declines in a ($\log \bar{y}$, $\log x$) grid.

Constant A represents the inputs to manufacture the first standard unit or quantity with new capital equipment embodying the newest technology. Because of technological progress, A has the tendency to decrease with time as the capital coefficient in manufacturing has the tendency to increase. If A is a function of time, it can be written as A(t).

Then we have:

A(0) = inputs of capital equipment at $t = 0$ just after the first standard unit or quantity is manufactured. From that point in time onwards this piece of capital equipment is termed "Defender".

A(T) = inputs of the newest capital equipment at point T in time when the "Defender" becomes obsolete. The newest piece of capital equipment is termed "Challenger".

Replacement of a Defender by a Challenger will occur at $t = T$ when:

$$\bar{y}(T) = A(0).x^{b} = A(T) \tag{29}$$

From (29) it follows that:

$$b = \frac{\ln\{A(T)/A(0)\}}{\ln x} \quad \text{for } 0 < (-b) < 1 \text{ and } A(0) > A(T) \tag{30}$$

which is the progress elasticity reflecting also the negative elasticity in the productivity growth function because $\bar{y}(T)$ is the reciprocal of productivity: input divided by output in time interval (0,T). If so, b may be identical to $-H_W(t^*)$, the integrated hazard at $t = t^*$ which is also an elasticity as described in Section VI.2.1.1. Then we obtain:

$$-H_W(t^*) = \frac{\ln\{A(T)/A(0)\}}{\ln x} \tag{31}$$

Since $S_W(t^*) = \exp[-H_W(t^*)]$, it follows from (31) that:

$$S_W(t^*) = [A(T)/A(0)]^{1/\ln x} \tag{32}$$

When (26) and (32) are compared, the concept seems to be similar. This is supported by empirical findings as published by NADLER & SMITH (1963). From their investigations it can be derived that the progress elasticity b ranges from $-0.3679 < b < -0.1948$ which corresponds for $-b = H(t^*)$ to $0.1948 < H(t^*) < 0.3679$. Since $H(t^*) = 1/\beta_3$, it follows that the shape parameter of the WEIBULL core distribution ranges from:

$$2.718 < \beta_3 < 5.134$$

The lower value, $2.718 \approx e$, coincides with the minimum value of:

$$t^*/\mu_3 = \beta_3^{-1/\beta_3} = H_W(t^*)^{H_W(t^*)}$$

found in Section II.6., Figure 2. The range is in agreement with nearly all β_3-estimates found in Chapter IV and summarized in Table IV-11.

When characteristic shape parameter $\beta_3^*[2] = 3.2589$ (related to a WEIBULL distribution of which mode and median are equal), it follows that:

$$b = -1/3.2589 = -0.3069$$

which gives a progress ratio of: $2^{-0.3068} = 0.8084$
The latter value is fully in agreement with the "80%-experience curve" derived by the BOSTON CONSULTING GROUP (1970).

Here again a strong relationship between the elasticities concerned is suggested. The reason for that relationship may be the underlying hazard process that they have in common. The negative progress elasticity of the L.C. and the capital elasticity in the COBB-DOUGLAS production function may be equal to the integrated hazard at the characteristic lifetime. It was proved that the latter is indeed a capital elasticity.

In respect of the above GULLEDGE & WOMER (1990) developed a dynamic cost model which combines a production function of the COBB-DOUGLAS type with the L.C. hypothesis applicable to the airframe industry for made-to-order production. The model describes the time paths of resource use (stock of knowledge and of capital) and production rate. They demonstrated that the knowledge/output ratio, α, attains its optimal value when (in original notations):

$$\alpha = l(t)/q(t) = \{\delta/(1-\delta)\}^{\frac{1}{2}} \qquad (33)$$

where δ is the learning curve parameter analogous to $-b$ in (27), $l(t)$ the rate of knowledge and $q(t)$ the output rate at time t. The optimal value of α satisfies the conditions for minimizing the required resources. High values of δ make resources more productive for a given stock of knowledge but the value of α is decisive for minimizing the required resources. If δ is replaced by $-b = 1/\beta_3$, very realistic values for α and β_3 are found on the basis of resource policy simulations carried out by GULLEDGE & WOMER. In any case, the subject discussed in the two subsections above is a matter of interest for further research.

VI.3. Summary and Findings

For the purpose of this study, a capital asset is defined as a tangible operating production or service system, being a manufactured investment good and a depreciable and reproducible component of capital stock. They are, generally, repairable and maintainable as they can be replaced partly or completely. Replacement at the end of their life, however, is not a condition as such.

Lifetime is defined as the length of the utilization period from the initial start of fulfilling a capital asset's production or service function until it is unfit for further use and discarded from the class of stock in question.

The objective was to extend our knowledge of lifetime characteristics of previously defined capital assets, and to facilitate modelling of their lifetime distributions to be used mainly as a complementary tool in economics and econometrics. The problem of the lack of data and information on life characteristics is the main reason why modelling of lifetime distributions for capital assets is meaningful.

The aim of capital assets or manufactured durables is to accomplish their production and/or survice function in the most economic manner. Optimally, marginal revenues are equal to marginal costs, the discarding rule. Within an interval of time, this rule involves the ratio of the value of goods or services produced (total output) to the value of the resources consumed (total input). In this view the discarding rule is related to productivity (output/input ratio). In Chapter II a productivity based indicator termed "Performance Rate" is defined as efficiency times effectiveness, equal to the ratio of real productivity to standard productivity. The latter is a time-dependent ratio which reflects the development of technology with time. Technological progress is in many cases the main driving force of the discarding process. The state-of-the-art in technology has also a lifetime which is a random variable that affects the Performance Rate. The productivity/performance-rate that yields sound definitions, was chosen as a point of departure in justifying discard due to economic obsolescence.

In Chapter II the "average capital consumption" concept was introduced which is the amount of capital consumed over the effective lifespan. Capital consumption is the consequence of the initial investment and also of the need to maintain a capital asset in a competitive condition in order to satisfy a specified performance rate. Hence, the cumulative amount of capital consists of two quantities, the (net, fixed) initial investment and the maintenance quantity. In this respect maintenance involves more than repair and/or restoration to the initial physical state. For us the maintenance quantity is an economic provision that serves as a counterbalance to compensate for a decreasing probability of survival with time.

An "average capital consumption" function (II/4) was derived which is represented by a U-shaped curve in a plot of cumulative capital consumption per unit of time versus lifetime. From this function it appeared that the minimum average total capital consumption per unit of time is equal to the current maintenance provision per unit of time when $t = t^*$. According to (II/5) the marginal (incremental) consumption or source depletion equals the long-term average total capital consumption or depletion when this amount is reduced to a minimum. That point in time, t^*, was defined as "characteristic lifetime".

It was stressed that there is an equilibrium resulting from a continuous process of deliberation about maintaining or discarding. Mathematically, this was written as an equality (II/6) on the basis of a probabilistic approach. This approach offered the opportunity to derive the basic average capital consumption formula (II/9). It appeared that the average total capital consumption is proportional to the initial investment and inversely proportional to the lifetime variable and the probability of survival. When the survival function is the result of an increasing hazard rate function with time, the average total capital consumption function is represented by a U-shaped curve. Its minimum value is attained at the characteristic lifetime t^*.

As a working hypothesis the WEIBULL distribution was chosen because of its hazard properties which allow a decreasing, a constant (time-independent) or an increasing hazard-rate function of time. It was then easy to derive the:

- characteristic lifetime t^* (II/30)
- integrated hazard at t^*: $H_W(t^*) = 1/\beta_3$ (II/31)

when β_3 is the (core) WEIBULL shape parameter.

It was demonstrated that the ratio t^*/μ (characteristic lifetime t^* to the WEIBULL size parameter μ) is minimized when the shape parameter $\beta = e \approx 2.7183$, identical if $\beta = 2$ or $\beta = 4$, and nearly constant for $1.6 < \beta < 6$. A fair approximation may be $t^* \approx 0.7\ \mu$. The characteristic lifetime in the case of an EXPONENTIAL distribution ($\beta = 1$) proved to be $t^* = \mu$.
In Chapter II it was shown that the hazard-rate function is, according to its definition, a conditional rate that is proportional to the maintenance need at a given point in time, and inversely proportional to the total amount of capital consumed up to that point in time. This remarkable result was obtained, no matter which (appropriate) lifetime distribution model is considered (II/39). Next it was demonstrated that the hazard rate at the characteristic lifetime is the reciprocal of that point in time (II/41). Evidence of that important finding was given for a WEIBULL distribution (II/42) which implies that $h_W(t^*) = 1/t^*$. For $1.6 < \beta < 6$ it applies that $h_W(t^*) \approx 1.43/\mu$ (II/43).

In Chapter III our lifetime model was constructed on the basis of a 3-component (composite) WEIBULL concept. It starts from a given population of one class (a set) of capital assets which have an identical production or service function and which operate independently. This population mass is exposed to three different modes of life-attacking processes resulting in three risk-specific hazard rates. Hence, the population mass is regarded as a 3-component collection. Each unit of the mass is predestined to fail due to a randomly selecting hazardous process that brings forth subpopulations. Subpopulation I fails during Phase I characterized by a decreasing hazard rate. Subpopulation II which has survived Phase I, fails during Phase II solely due to sudden change disruptions characterized by a constant hazard rate (time-independent). Subpopulation III fails solely because of an increasing hazard rate due to economic aging and technical wear and tear.
The duration of Phase I in time interval $(0,t)$ is $t = 1$; thus the time scale of our model consists of Phase I-units of time. The fundamental characteristic of our 3-component (composite) lifetime distribution of lifetimes is the relationship between its parameters. The integrated function is represented by a continuous curve on which the Phase I and II

hazards are equal at their partition I/II. Similarly the integrated hazard at $t = a$ is continuous, with the Phase II and III hazards equal at that time. From these points of departure it was easy to construct a 3-component (composite) WEIBULL distribution model with 7 parameters (III/1). With the 3 restrictions, the number was reduced to 4 parameters (III/2 to III/8). One of the restrictions is the consequence of the assumption that a linearly increasing hazard rate is a limiting case for a progressively increasing aging and wear and tear process during Phase III. Thus the lower threshold value of the shape parameter of the WEIBULL core distribution for Phase III amounts to $\beta_3 = 2$.

We have theoretically derived the following 3 characteristic shape parameters for Phase III:

$$\beta_3^*[1] = 3.05,\ \beta_3^*[2] = 3.667 \text{ and } \beta_3^*[3] = 3.2589$$

More precisely, $\beta_3[3] = 1/(1 - \ln 2)$ which applies when mode and median of a WEIBULL distribution are identical.

At the end of Chapter III the graphical form of our model is shown as a plot of $\{\ln H(t)\}$ versus $(\ln t)$, a set of 3 straight lines representing the distinctive and successive survivor curves of Phase I, II and III respectively (Figure 6). This triangle-based concept is a graphical reflection of the findings obtained in Chapter III.

In Chapter IV the model elaborated in Chapter III was tested by means of 96 sets of empirical retirement data derived from 4 main sources including 65 sets of the original WINFREY data. In addition the 18 WINFREY type curves were analyzed in the light of our model. It was established that 3 of his 6 left-modal curves (L^0, L^1 and L^3) are fair approximations of a WEIBULL survivor curve. Type curve L^2 agrees quite well with a WEIBULL survivor curve with $\beta = 2.741$.

Three (S^1, S^2 and S^3) of his 7 symmetrical type curves are good approximations of WEIBULL survivor curves. WINFREY's right-modal R-type curves agree more or less with a two or 3-component (composite) distribution, however, it appeared that our modelled survivor curves fit the empirical data better than those constructed by WINFREY.

Almost all sets of empirical retirement refer to individuals which data are in some degree aggregated groups or categories or kind of capital assets and manufactured durables. The raw data of individuals are, generally, poorly documented. The problem of heterogeneity and inaccurate measurement was solved by assuming a homogeneous mass in which empirical retirement data of individuals are erroneously measured and recorded. The effect of errors of measurement was simulated to gather insight into the consequences. The result on the basis of a set of hypothetical WEIBULL distributions was that normally distributed measuring errors have a negligible impact on the value of the size parameter. But the value of the shape parameter is significantly reduced as the relative measurement errors increase and the size parameter decreases. When a regression technique is applied for parameter estimation, the coefficient of determination remains high if the measurement errors are limited to roughly 10% of the associated mean value. Measurement errors in combination with data grouping results in a high coefficient of determination.

The testing procedure was a combination of graphical and analytical techniques starting with the determination of the nonparametric estimate of the integrated hazard according to the KAPLAN-MEIER method. A graphical representation of the data points in a $\{\ln H(T_j)\}$, $(\ln t)$ plot was used to segregate data points of Phase I, II or III. Then the WEIBULL parameters of the Phase III survivor curve were estimated by means of the ML-method and by means of a quasi-linear regression technique. Subsequently, it was checked which set of parameter estimates was preferred. This was done in two ways; analytically by determining the discrepancies between the empirical and the estimated probability of survival values, and graphically by plotting the H-residuals to ensure that the WEIBULL concept is not misspecified by the neglect of random heterogeneity in the hazard function. In 30 representative and selected cases it appeared that the fit of our model to the Phase III data points was always acceptable; misspecification was rejected in all cases. The KM-method of parameter estimation followed by a quasi-linear regression technique gave reasonable results in all cases and performed better in 20 out of 30 cases. The ML-method of parameter estimation gave slightly better results in the remaining 10 cases.

Phase II appeared in 13 out of the 30 sets. Curve fitting of an EXPONENTIAL distribution to the Phase II data points resulted in low valued discrepancies between the empirical and the estimated probability of survival fractions. The plots of H-residuals showed good results; thus misspecification was rejected in all cases, except possibly for Code 38-1.

Phase I was met in only one case that concerned with Automobiles 1900-1922 (Code 62-2).

The theoretically derived values of the WEIBULL shape parameter for the core (Phase III) distribution correspond with the values which were empirically derived. The range $1.6 < \beta_3 < 6$ coincides with the findings except for some primitive goods such as loading coils and crossties. Generally, the β-estimates are close to the theoretically derived characteristic values of the WEIBULL shape parameter.

It was found that the size parameter of the EXPONENTIAL distribution related to Phase II was underestimated and thus the partition parameter $\hat{a}$ was overestimated when compared with the values obtained by our model. Generally, the duration of Phase I was longer than expected on the basis of our model. However, the differences between the empirical findings concerned with Phases I and II and our model are negligible in terms of discrepancies in probability of survival fractions. As concluded in Section IV.9 above, the validity of our probabilistic lifetime model on the basis of a 3-component (composite) WEIBULL distribution was demonstrated. Misspecification was rejected on the basis of the restrictive testing criteria as specified.

In Chapter V we elaborated a depreciation methodology based on the principles of a probabilistic service lifespan of capital assets and manufactured durables. For this purpose depreciation is defined as an economic provision to compensate for the decrease in value as a consequence of declining productivity as reflected by the performance-rate pattern over time. Cumulative depreciation is identical to capital consumption C(t) as defined above (II/2) in Chapter II. It was postulated that the ratio of the cumulative depreciation (capital consumption) and the amount to be depreciated is identical to the integrated hazard as a function of time (V/1). It was shown that the cumulative depreciation is a function of time proportional to the initial investment times H(t).exp[H(t)]. Since the latter product is monotonically increasing with time, the cumulative amount to be depreciated can/will exceed the initial investment. This product is one when $H_w(t) = S_w(t) = 0.56714$ which is shortly before the average lifetime which agrees with the results of operative depreciation methodologies.

Since the cumulative depreciation ratio, $D(t)/C(t)$, is identical to the integrated hazard, the continuous pattern differs for differently valued parameters related to Phase I, II and III characteristics. For this purpose a generalized depreciation ratio function (V/9) was derived. The quantity "one minus the integrated hazard" was defined as the "net value ratio" which is equal to ESTEBAN's elasticity function (II/27) times the integrated hazard at the characteristic lifetime $H(t^*)$. As stressed in Section VI.2.1. above, $H(t^*)$ may be regarded as a capital elasticity that reflects the rate of change of the "net value ratio" to the rate of change in the interval $(t, t+dt)$ of the p.d.f. of lifetimes. Because $H_W(t^*) = 1/\beta_3$, the capital elasticity meant here is the inverse WEIBULL shape parameter of the Phase III core distribution.

For the ratio $D(t)/C(t)$ and $C(t)/I$ typical values are derived for the mode, the median and the size parameter (V/17 to V/25). The WEIBULL size parameter related to Phase III plays an essential role in capital consumption because at that point in time the ratio $D(\mu_3)/C(\mu_3) = 1$ and ESTEBAN's p.d.f. elasticity is zero. Time interval $(0,\mu_3)$ corresponds to the pay-off period P as defined by TERBORGH (1949/1958) and discussed in Section VI.1. above (VI/3 to VI/8). The crucial value of $H(\mu_3)$ was also demonstrated in the light of technological progress.

We have regarded technological progress as an integer counting process resulting from learning, practicing and creative searching. That is, innovations are highly localized events in a time continuum which is characteristic of a stochastic point process. Assuming that the number of innovative events are distributed according to a non-homogeneous POISSON distribution, it was shown that the probability of one fatal innovation attains its maximum value of $\exp[-1]$ when the integrated hazard is one. When we have a WEIBULL distribution of lifetimes, including those of the state-of-the-art in technology, the point $t = \mu_3$ is the most likely moment for attaining $D(\mu_3)/C(\mu_3) = 1$.

The ratio $C(\mu_3)/C(t^*) = \exp[1 - (1/\beta_3)] = 2$ when $\beta_3 = 1/(1 - \ln 2)$, is an essential finding because then $\beta_3 = \beta_3^*[3] = 3.2589$ (V/28). The cumulative capital consumption at $t = \mu_3$ is twice as much as at $t = t^*$ which is plausible.

A firm will continue operating an existing asset until its operating costs equated the market price because it would maximize the net present value of that asset. The point of equilibrium is the optimum lifetime which was determined according to DE LA MARE (1982). Assuming that this point agrees on average with $t = \mu_3$, the size parameter μ_3 can be estimated. It appeared that μ_3 is proportional to the logarithmic initial productivity, and inversely proportional to the rate of decline in performance p.a. when μ_3 is expressed in years. The rate of decline in performance is the sum of the rate of technological progress p.a. and the rate of decline in production efficiency p.a. as the manufacturing asset becomes older. Here again we meet the strong connection of the size parameter with technological progress and with the initial productivity say at $t = 1$ when Phase I is experienced. The findings agree well with the fundamentals of relevant replacement models discussed in Section VI.1. above.

It was shown that the depreciation ratio pattern is governed by the WEIBULL shape parameter (Figure 15). This pattern is similar to the one developed empirically by the US DEPARTMENT OF LABOR & BUREAU OF STATISTICS. The assumed pattern of depreciation is expressed by a flexible function such that convex, linear and concave patterns can be generated. The moment at which the net value becomes zero is regarded as the mean of a lifetime variable distributed according to a vertically or horizontally truncated NORMAL distribution. It was demonstrated that the cumulative depreciation ratio pattern of that model matches up significantly with our model with regard to Phase III (Figure 16). An appropriate parametric relationship between the two models was determined (V/39). Furthermore, it transpired that discard frequencies generated by this concept show an almost perfect WEIBULL distribution for Phase III..

Loss of value in road pavements is identical to the capital consumption associated with maintaining the function of a road in the most economic manner. Maintenance in this respect is more than repairs; it also includes upgrading to the latest standards to fulfil traffic requirements. The underlying process of deterioration of roads and pavements is analogous to that of capital equipment and manufactured durables, since deterioration is the consequence of wear and tear and of the newest needs. In the case of pavements the economic consequences are upgrading and maintenance costs equal to depreciation. Therefore, a maintenance cost and planning model for pavements may be equivalent to a depreciation model. It was demonstrated that such an empirical model developed by KONING and MOLENAAR (1987) is indeed identical to our depreciation model, with regard to Phase III. Furthermore, it was demonstrated that this road pavement model has a robust theoretical basis when the principles of probabilistic fracture mechanics and crack propagation under cyclic loading are taken into consideration.

Finally, an amortization/depreciation concept was elaborated for Phase III. This concept is closely related to the rental price of capital services which is the same as depreciation or utilization of capital. Therefore the principles of leasing are applicable. The capital for productive equipment is provided to fulfil a production or service function in the most economic manner. The rental (lease) price of capital includes maintenance such that there is perfect substitution between new and older assets. This implies that competitive performance is ensured at every point in time. The value is equal to the capitalization of a future income stream arising from fixed amounts at regular intervals. Each fixed amount (installment) covers an interest proportion (decreasing with time) on the principal outstanding, and a proportion (increasing with time) on capital consumption. It was demonstrated that the cumulative depreciation (capital consumption) curves generated by the amortization model are almost identical to the ones generated by our basic model applied to the core WEIBULL lifetime distribution (Figure 17). The parametric relationship (V/50) is robust because both models are based on productivity, which in this case is equivalent to profitability. The advantage of the amortization/depreciation model is that one of the parameters is a straightforward representation of the profitability;

the other represents the number of fixed amounts of income at regular intervals and must be identical to the size parameter of the core WEIBULL lifetime distributions of capital assets.
On the evidence of a perfect substitution of new and older depreciable and reproducible assets or manufactured durables at every point in time, it was shown that cumulative capital consumption (in constant prices) is not limited to the initial investment. The amount (including the initial investment) to be depreciated increases with time and is related to the reciprocal of the probability of survival. That amount is not the same as that which, in accounting practice, is referred to as "replacement value".

In Section VI.1. of this chapter it appeared that the weakness of deterministic lifetime models is that as many parameters can/will vary over time, these changing values lead to different lifetimes. Such models are needed and useful but it may be wise to accept that lifetime is a stochastic variable resulting in a probabilistic lifetime model which provides for successive and distinctive life phases with differently valued parameters. That is what was attempted in this study.

REFERENCES

ABERNATHY, W.J. and WAYNE, K. (1974), Limits of the Learning Curve. Harvard Business Review, pp. 109-119, Harvard Business School Publishing Division, Boston, USA.

AITKIN, M. and CLAYTON, D. (1980), The Fitting of Exponential, Weibull and Extreme Value Distributions to Complex Censored Survival Data using GLIM. Applied Statistics, Vol. 29, No. 2, pp. 156-163, Centre for Applied Statistics, University of Lancaster, GB.

ARROW, K.J. (1962), The Economic Implications of Learning by Doing. Review of Economic Studies, Vol. XXIX(3), No. 80, pp. 155-173, Edinburgh, Scotland.

ASHER, H. and FEINGOLD, H. (1984), Repairable Systems Reliability, Modeling, Inference, Misconceptions and their Causes. ISBN 0-8247-7276-X, Marcel Dekker, Inc., New York, USA.

ASTM (1981), Special Technical Publication of approved papers, Symposium on Probabilistic Methods for Design and Maintenance of Structures. ASTM-STP 798, PCN 04-798000-39, 1983, Philadelphia, USA.

BARLOW, R.E. and MARSHALL, A.W. (1965), Tables of Bounds for Distribution with Monotone Hazard Rate. The American Statistican, Vol. 60, pp. 872-890, The American Statistical Association, Washington, D.C., USA.

BARNA, T. (1957), The Replacement Cost of Fixed Assets in British Manufacturing Industry in 1955. Journal of the Royal Statistical Society, Vol. 120, Part I, pp. 1-36, London, GB.

BARNA, T. (1959), Measuring Capital and the Length of Life of Capital Goods. Cahiers Economiques de Bruxelles, No. 5, pp. 71-81, Belgium.

BARNA, T. (1961), On Measuring Capital. UUTZ, F.A. and HAGUE, D.C. (Ed.): The Theory of Capital, pp. 75-94, London, GB.

BEKKER, P.C.F. (1980), Lifetime of Dwellings and Scarce Resources. Ph.D. thesis, Slovak University of Technology, Bratislava, Czechoslovakia.

BIØRN, E., HOLMØY, E. and OLSEN, O. (1989), Gross and Net Capital, and the Form of the Survival Function: Theory and some Norwegian Evidence. Review of Income and Wealth, Series 35, No. 2, pp. 133-149, GB.

BOSTON CONSULTING GROUP (1970). Perspectives on Experience, Boston, Massachusetts, USA.

CBS (1982), Replacement value, Repair and Economic Lifetime of mechanically operated Tools. Report E12-82-49, BPA-number H.830-82-E12, January 1982, Central Bureau of Statistics, Heerlen, Netherlands.

CBS (1987), Dalcy (database lifetime-cycle of manufacturing assets), Central Bureau of Statistics, Heerlen, Netherlands.

CHESHER, A. and IRISH, M. (1987), Residual Analysis in the Grouped and Censored Normal Linear Model. Journal of Econometrics, Vol. 34, pp. 33-61, North-Holland, Netherlands.

CHOW, G.C. (1960), Demand for Automobiles in the United States, a study in consumer durables, North-Holland Publishing Company, Amsterdam, Netherlands.

COBB, C.W. and DOUGLAS, P.H. (1928) A Theory of Production. American Economic Review, Papers and Proceedings, 1928; DOUGLAS, P.H. The Theory of Wages, 1934; Are there Laws of Production? American Economic Review, American Economic Association, Nashville, USA.

COX, D.R. and SNELL, E.J. (1968), A General Definition of Residuals. Journal of the Royal Statistical Society, B 30, London, GB.

COX, D.R. and OAKES, D. (1984), Analysis of Survival Data. ISBN 0-412-21910-7, Chapman & Hall, London, GB., and New York, USA.

CRAMER, J.S. (1958), The Depreciation and Mortality of Motor Cars. Journal of the Royal Statistical Society, Vol. 121, Part I, pp. 18-59, London, GB.

CRAMER, J.S. (1969), Empirical Econometrics, ISBN 0-7204-3050-X, pp. 10-11, North-Holland Publishing Company, Amsterdam, Netherlands.

DEMING, W.E., and SINGPURWALLA, N.D. (1989), A Bear Trap in Using the Iowa-Curve Methodology for Property Retirements and Depreciation Charges. The American Statistician, Vol. 43, No. 1, pp. 12-16, American Statistical Association, Washington, D.C., USA.

DIJKMEIJER, V. (1990), Simulation of Inaccuracy on the Basis of Hypothetical Weibull Distributions. B.Sc.-stage work, Polytechnic (HTS), 's-Hertogenbosch, Netherlands.

EDOSOMWAN, J.A. (1987), Integrating Productivity and Quality Management, ISBN 0-8247-7657-7, HD56.E28, Marcel Dekker, Inc., New York, USA.

ESTEBAN, J. (1978), A General Density Function for the Distribution of Income, Revised Version, Barcelona, Spain.

ESTEBAN, J. (1986), Income-share Elasticity and the Size distribution of Income. International Economic Review, Vol. 27, No. 2, pp. 439-444, University of Pennsylvania, Philadelphia, USA.

FELDSTEIN, M.S. and FOOT, D.K. (1971), The Other Half of Gross Investment: Replacement and Modernization Expenditures. Review of Economics and Statistics, Vol. 53, No. 1, pp. 49-58, The President and Fellows of Harvard College, North Holland Publishing Company, Amsterdam, Netherlands.

FELDSTEIN, M.S. and ROTHSCHILD, M. (1974), Towards an Economic Theory of Replacement Investment. Econometrica, Vol. 42, No. 3, pp. 393-423, Department of Economics, Princeton University, New Yersey, USA.

GRIFFIN, T.J. (1975), Revised Estimates of the Consumption and Stock of Fixed Capital. Economic Trends, No. 264, pp. 126-129, HM's Statistical Office, London, GB.

GROES, N. (1976), Measurement of Capital in Denmark. Review of Income and Wealth, 22, No. 3, pp. 271-286, GB.

GULLEDGE, T.R. and WOMER, N.K. (1990), Learning Curves and Production Functions: an Integration. Engineering Costs and Production Economics, No. 20 (1990) 3-12, Elsevier Science Publishers, Amsterdam.

HICKS, J.R. (1965), Capital and Growth, Oxford University Press, London, GB.

HINSKENS, P. and VAN WIERINGEN, E. (1989), Investigation on Lifetime Distributions of Capital Assets. M.Sc.-thesis work conducted by BEKKER, P., and DAAMS, C., Department of Industrial Engineering and Management Science, Eindhoven University of Technology, Eindhoven, Netherlands.

HIRSCH, W.Z. (1956), Firm Progress Ratios. Econometrica, Vol. 24, pp. 136-143, Department of Economics, Princeton University, New Yersey, USA.

HULST, W.G.H. van (1973), The Replacement of Durable Equipment, Ph.D.-thesis, ISBN 90-207-0399-4, Stenfert Kroese, Leiden, Netherlands.

JOHNSON, N.L. and KOTZ, S. (1970), Distributions in Statistics: Continuous univariate Distributions, Vol. 1, John Wiley and Sons, New York, USA.

JORGENSON, D.W. (1965), Anticipations and Investment Behavior in the Brookings Quarterly Econometric Model of the United States. Rand McNally and Company, Chicago, USA.

JORGENSON, D.W. (1974), The Economic Theory of Replacement and Depreciation, SELLEHAERTS, W. (Ed.). Econometrics and Economic Theory, Essays in honour of Jan Tinbrgen, pp. 189-221, MacMillan Press, London, GB.

KAERGAARD, N. (1970), Efterspørgelstrukturen for personbiler i Danmark. Nationaløkonomisk Tidsskrift, DK.

KALBFLEISCH, J.D. and PRENTICE, R.L. (1980), The statistical Analysis of Failure Time Data. ISBN 0-471-05519-0, John Wiley and Sons, New York, USA.

KAO, J.H.K. (1959), Graphical Estimation of Mixed Weibull Parameters in Life-Testing Electron Tubes. Technometrics, Vol. 1, No. 4, pp. 389-407, Princeton, USA.

KAPLAN, E.L. and MEIER, P. (1958), Nonparametric Estimation from incomplete Observations. Journal of the American Statistical Association, Vol. 53, No. 282, pp. 457-481, Menasha, USA.

KENDALL, M. and STUART, A. (1979), The Advanced Theory of Statistics. Vol. 2-1979, pp. 465-484, IBSN 0-85264 239 3, Charles Griffin & Comp., London.

KIRNER, W. (1968), Zeitreihen für das Anlagevermögen der Wirtschaftsbereiche in der Bundesrepublik Deutschland. DIW-Beitrage zur Strukturforschung, Heft 5, Duncker und Humblot, Berlin, FRG.

KONING, P.C. and MOLENAAR, A.A.A. (1987), Pavement Management System for Municipalities with Emphasis on Planning and Cost Models. Proceedings of the Sixth International Conference Structural Design of Asphalt Pavements, Vol. 1, pp. 1041-1049, University of Michigan, Ann Arbor, Michigan, USA.

KRAKER, A. de, TICHLER, J.W. and VROUWENVELDER, A.C.W.M. (1982), Safety, Reliability and Service Life of Structures. Heron, Vol. 27, No 1, Department of Civil Engineering, Delft University of Technology, Delft and TNO-Institute for Building Materials and Structures, Rijswijk (ZH), Netherlands.

LANCASTER, T. (1985), Generalised Residuals and Heterogeneous Duration Models; With Applications to the Weibull Model. Journal of Econometrics, Vol. 28, pp. 155-169, North-Holland, Netherlands.

LANCASTER, A.D. (1990), Econometric Analysis of Transition Data. ISBN 0-521-26596-7, Cambridge University Press, Cambridge, GB.

LAWLESS, J.F. (1982), Statistical Models and Methods for Lifetime Data, ISBN 0-471-08544, John Wiley and Sons, New York, USA.

LIPSEY, R.G., STEINER, P.O. and PURVIS, D.D. (1984), Economics. ISBN 0-06-350425-1, 7th Edition, International Edition, Harper and Row, New York, USA.

LüTZEL, H. (1971), Das Reproduzierbare Anlagevermögen in Preisen von 1962. Wirtschaft und Statistik, Heft 10, pp. 593-610, Stuttgart, FRG.

LüTZEL, H. (1972), Das Reproduzierbare Sachvermögen zu Anschaffungs- und zu Wiederbeschaffungspreisen. Wirtschaft und Statistik, Heft 11, pp. 611-694, Stuttgart, FRG.

LüTZEL, H. (1976), Altersaufbau des Anlagevermögens. Wirtschaft und Statistik, Heft 4, pp. 217-221, Stuttgart, FRG.

MALCOMSON, J.M. (1975), Replacement and the Rental Value of Capital Equipment Subject to Obsolescence. Journal of Economic Theory, No. 5, pp. 24-41, Academic Press, Inc., New York, USA.

MANN, N.R., SCHAFER, R.E. and SINGPURWALLA, N.D. ((1974), Methods for Statistical Analysis of Reliability and Life Data. ISBN 0-471-56737, John Wiley and Sons, New York, USA.

MARE, R.F. de la (1982), Manufacturing Sytems Economics, ISBN 0-03-910363-3, pp. 427-464, Holt, Rhinehart and Winston, Ltd., Eastbourne, East Sussex, Ireland.

MCDONALD, J.B. (1984), Some Generalized Functions for the Size Distributions of Income. Econometrica, Vol. 52, No. 3, Department of Economics, Princeton University, New Yersey, USA.

NADLER, G. and SMITH, W.D. (1963), Manufacturing Progress Functions for Types of Processes. International Journal of Production Research, Vol. 2, pp. 115-135, Taylor and Francis, London, GB.

NATESAN, J. and JARDINE, A.K.S. (1986), Graphical Estimation of Mixed Weibull Parameters for Ungrouped Multicensored Data: Its Application to Failure Data. Maintenance Management International, Vol. 6, pp. 115-127, Elsevier Science Publishers B.V., Amsterdam, Netherlands.

NELSON, W. (1969), Hazard Plotting for Incomplete Failure Data. Journal of Quality Technology, Vol. 1, No. 1, pp. 27-52, American Society for Quality Control, Milwaukee, USA.

NEWBY, M.J. (1980), The Properties of Moment Estimators for the WEIBULL distribution Based on the Sample Coefficient of Variation. Technometrics, Vol. 22, No. 2, May 1980.

NEWBY, M.J. (1987), Failure Models and Data Analysis based on Ageing Properties. International Journal of Quality and Reliability Management, Vol. 4, No. 3-1987.

NEWBY, M.J. (1991), Estimation of PARIS-ERDOGAN Law Parameters and the Influence of Environmental Factors on Crack Growth. International Journal of Fatigue (Materials, Sructures, Components), Vol. 13-1991, Butterworth-Heinemann, Guildford.

NICKELL, S. (1975), A Closer Look at Replacement Investment. Journal of Economic Theory, Vol. 10, No. 1, pp. 54-88, Academic Press, Inc. New York, USA.

PARIS, P.C. and ERDOGAN, F. (1963), Transactions. American Society of Mechanical Engineers, Vol. 85, Series D (4), pp. 528-534, New York, USA.

PRAAGMAN, J. (1986), Efficiency of Change Point Tests. Ph.D.thesis, Eindhoven University of Technology, Eindhoven, Netherlands.

REDFERN, P. (1955), Net Investment in Fixed Assets in the United Kingdom, 1938-1953. Journal of the Royal Statistical Society, Vol. 118, pp. 141-192, London, GB.

REEKEN, A.J. van (1987), Begrippen rondom Kwaliteit. Bedrijfskunde, No. 2, jrg. 59, pp. 173-182, Kluwer, Deventer, Netherlands.

RIDDER, G. (1987), Life Cycle Patterns in Labor Market Experience. A statistical analysis of labor market histories of adult men. Ph.D.-thesis, University of Amsterdam, Department of Economics, Amsterdam, Netherlands.

SCHIFF, E. (1958), Gross Stock Estimated from Past Installations. The review of Economics and Statistics, North-Holland Publishing Company, Amsterdam, Netherlands.

SCHWORM, W.E. (1979), Tax Policies, Capital Use and Investment Incentives. Journal of Public Economics. No. 12, pp. 191-204, Elsevier Science Publishers, North-Holland, Amsterdam, Netherlands.

SINK, D.S. (1985), Productivity Management, Planning Measurement and Evaluation, Control and Improvement. ISBN 0-471-89176-2, John Wiley and Sons, New York, USA.

SMIT, H.P. (1982), The World Rubber Economy to the Year 2000. Ph.D.-thesis, Free University, Amsterdam, Netherlands.

SMITH, V.L. (1961), Investment and Production: A study in the Theory of the Capital Using Enterprise. Harvard University Press, Cambridge, Massachussetts, USA.

SON, K.S. and PARK, C.S. (1987), Economic Measure of Productivity, Quality and Flexibility in Advanced Manufacturing Systems. Journal of Manufacturing Systems, Vol. 6, No. 3, Society of Manufacturing Engineers, Michigan, USA.

STACY, E.W. (1962), A Generalization of the Gamma Distribution. Annals of Mathematical Statistics, 33, pp. 1187-1192, Baltimore, USA.

STEWART, W.T. (1978), A Yardstick for Measuring Productivity. Industrial Engineering, Vol. 10, No. 2, pp. 34-37, Institute of Industrial Engineers, Norcross, Atlanta, USA.

SUMANTH, D.J. (1987), Productivity Management Frontiers I. ISBN 0-444-42741-4, Elsevier Science Publishers, Amsterdam, Netherlands.

TARIGAN, T.E. (1985), The Role of Quality Control in Tyre Maintenance at HTM, Fieldwork Report KW/EZ/070585, Research Institute for Management Science (RVB), Maastricht, Netherlands.

TENGBLAD, A. and WESTERLUND, N. (1976), Capital Stock Consumption Estimates by Industries in the Swedish National Accounts. Review of Income and Wealth, 22, No. 4, pp. 331-344, GB.

TERBORGH, G. (1949), Dynamic Equipment Policy. McGraw-Hill, New York, USA.

TERBORGH, G. (1958), Business Investment Policy: a MAPI study and Manual, Lord Baltimore Press, Washington, D.C., USA.

U.S. BUREAU OF LABOR STATISTICS (1979), Capital Stock Estimates for Input-Output Industries: Methods and Data. U.S. Department of Labor, Bulletin 2034, Bureau of Labor Statistics, Washington, D.C., USA.

VERGES-ESCUIN, R. (1981), The Measurement of Dwelling Stock Depletion in Forecasting Housing Needs. The Delegation of the International Union of Architects, Paris. United Nations Seminar on the Forecasting and Programming of Housing, March/April 1981, Paper HBP/SEM.27/COM./UIA.1., New York, USA.

VOORDOUW, J. (1981), Survivor Curve of Passenger Cars. Internal Report, Division of Registrated Stock of Vehicles, Central Bureau of Statistics (CBS), Voorburg, Netherlands.

WARD, M. (1976), The measurement of Capital: The Methodology of Capital Stock Estimates in OECD Countries. OECD-Publication, Paris, France.

WEIBULL, W. (1951), A Statistical Distribution Function of Wide Applicability. Journal of Applied Mechanics, pp. 293-297, New York, USA.

WINFREY, R.M.S. and KURTZ, E.B. (1931), Life Characteristics of Physical Property. IOWA State College of Agriculture and Mechanic Arts, Bulletin 103, Vol. XXX, No. 3, Ames, Iowa, USA.

WINFREY, R.M.S. (1935), Statistical Analysis of Industrial Property Retirements. IOWA Engineering Experiment Station, IOWA State College of Agriculture and Mechanic Arts, Bulletin 125, Vol. XXXIV, No. 28, Ames, Iowa, USA.

WRIGHT, T.P. (1936), Factors affecting the Cost of airplanes. Journal of Aeronautical Sciences, Vol. 3, p. 122, New York, USA.

YELLE, L.E. (1979), The Learning Curve: Historical Review and Comprehensive Survey. Decision Sciences, No. 10, p. 302, College of Business Administration, University of Florida, USA.

YELLE, L.E. (1983), Adding Life Cycles to Learning Curves. Long Range Planning, Vol. 16, No. 6, pp. 82-87, Pergamon Journals, Ltd. London, GB.

REVIEWED LITERATURE

ASHER, H. (1956), Cost-Quantity Relationships in the Airfame Industry. A Report prepared for United States Air Force Project Rand. The Rand Corporation, Santa Monica, California, USA.

ASTM (1983), Standard Practice for Measuring Life-cycle Costs of Buildings and Building Systems. E 917-83, ASTM-Headquarters, Philadelphia, USA.

BARLOW, R.E. and HUNTER, L. (1959), Optimum Preventive Maintenance Policies. Operations Research Society of America, Chapman and Hall, Washington, D.C., USA.

BAUMGART, E.R. and KRENGEL, R. (1970), Die Industrielle Vermögensrechnung des DIW. Beiträge zur Strukturforschung, Heft 10, Duncker und Humblot, Berlin, FRG.

BEKKER, P.C.F. (1982), A Life-Cycle Approach in Building. Building and Environment, Vol. 17, No. 1, pp. 55-61, Pergamon Journals, Ltd., London, GB.

BEKKER, P.C.F. (1983/1984), Economy and Building Growth. Annual Journal of the Building Technology Society, Vol. 10, Kualu Lumpur, Malaysia.

BEKKER, P.C.F. (1985/1986), Quality/Quantity in Building. Annual Journal of the Building Technology Society, Vol. 12, Kuala Lumpur, Malaysia.

BORSCH-SUPAN, A. (1986), Household Formation, Housing Prices and Public Policy Impacts. Journal of Public Economics, Vol. 30, No. 2, pp. 145-164, Amsterdam, Netherlands.

BOULDING, K.E. (1966), The Economics of the Coming Spaceship Earth, in Henry Jarret, (Ed.). Environmental Quality in a Growing Economy, pp. 121-132, John Hopkins Press, Baltimore, USA.

BREMS, H. (1967), Quantitative Economic Theory: A Synthetic Approach, John Wiley and Sons, New York, USA.

BUTTER, F.A.G. den (1976), De Optimale Economische Levensduur van Kapitaalgoederen in een Jaargangenmodel met een vaste Kapitaalcoëfficiënt. Maandschrift Economie, Vol. 7, pp. 396-405, Kluwer, Deventer, Netherlands.

BUTTER, F.A.G. den (1977), De Economische Levensduur van Kapitaalgoederen in een Clay-Clay Jaargangenmodel. Maandschrift Economie, 41, No. 5, pp. 205-231, Kluwer, Deventer, Netherlands.

CBS-Contribution to Statistics on 1985 Stock of Capital (1986). Staatsuitgeverij/CBS-publicaties, 's Gravenhage, Netherlands.

CHENERY, B. (?), Engineering Production Functions. Massaschussetts Institute of Technology, Quarterly Journal of Economics, pp. 507-531, John Wiley and Sons, New York, USA.

CHOW, G.C. (1957), Statistical Demand Functions for Automobiles and their Use in Forecasting, in A.C. Harbergen (Ed.), The Demand for Durable Goods, University of Chicago, Chicago, USA.

CLAROTTI, C.A. and LINDLY, D.V. (1988), Accelerated Life Testing and Experts' Opinion in Reliability, Course CII of the International School of Physics: Enrico Fermi, 1988, North-Holland Physics Publishing Company, Amsterdam, Netherlands.

COHEN, A.C. (1965), Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples. Technometrics, Vol. 7, No. 4, pp. 579-588, The American Society for Quality Control and the American Statistical Association, Washington, D.C., USA.

COX, D.R. (1962), Renewal Theory. ISBN 0-412-20570-X, Chapman and Hall, Ltd., New York, USA.

COX, D.R. and ISHAM, V. (1980), Point Processes. ISBN 0-412-21910-7, Chapman and Hall, Ltd., London, GB.

DALY, H.E. (1973), Towards a Steady-State Economy. ISBN 0-7167-0793-4, W.H. Freeman and Company, San Francisco, USA.

EISNER, R. (1972), Components of Capital Expenditures: Replacement and Modernization versus Expansion. Review of Economics and Statistics, Vol. 54, No. 3, pp. 297-305, the President and Fellows of Harvard College, North Holland Publishing Company, Amsterdam, Netherlands.

FINKELSTEIN, J.M. (1976), Confidence Bounds on the Parameters of the Weibull Process. Technometrics, Vol. 18, No. 1, pp. 115-117, The American Society for Quality Control and the American Statistical Association, Washington, D.C., USA.

FORRESTER, J.W. (1971), World Dynamics, Wright-Allen-Press, Cambridge, Massachussets, USA.

GEURTS, J.H.J. (1986), On the Selection of Elementary Maintenance Rules. Ph.D-thesis. Department of Industrial Engineering and Management Science, Eindhoven University of Technology, Eindhoven, Netherlands.

GLASSER, G.J. (1969), Planned Replacement: Some Theory and its Application. Journal of Quality Technology, Vol. 1, No. 2, pp. 110-119, America Society for Quality Control, Milwaukee, USA.

GRIFFIN, T.J. (1976), The Stock of Fixed Assets in the United Kingdom: How to make best Use of the Statistics. Economic Trends, No. 276, HM's Statistical Office, London, GB.

GROSS, A.J. and CLARK, V.A. (1975), Survival Distributions: Reliability Applications in the Biomedical Sciences. ISBN 0-471-32817-0, John Wiley and Sons, New York, USA.

HARTLER, G. (1981), Statistische Methoden für die Zuverlässigkeitsanalyse. VEG Verlag Technik Berlin, Berlin, FRG.

HENDRIKS, A. (1978), Een Keerpunt in de Produktie. ISBN 90-6275-027-3, University Press Delft, Delft, Netherlands.

HIBBERT, J., GRIFFIN, T.J. and WALKER, R.L. (1975), Development of Estimates of the Stock of Fixed Capital in the United Kingdom. 14th General Conference of the IARIW, Central Statistical Office, London, GB.

HILTON, B. (1986), Indefensible ? Accountancy, p. 123, Muusses, Purmerend, Netherlands.

HUTCHINSON, G.K. and SINHA, D. (?), Quantification of the Value of Flexibility. Journal of Manufacturing Systems, Vol. 8, No. 1, pp. 47-57, Society of Manufacturing Engineers, Michigan, USA.

KEMP, M.C. and KONDO, H. (1986), Overlapping Generations, Competitive Efficiency and Optimal Population. Journal of Public Economics, Vol. 30, No. 2, pp. 237-247, Amsterdam, Netherlands.

KENDRICK, J.W. and LEE, K.S. (1976), Quarterly Estimates of capital Stocks in the U.S. Private Domestic Economy, by Major Industry Group. Review of Income and Wealth, 22, No. 4, pp. 345-352, GB.

KIEFER, N.M. (1988), Economic Duration Data and Hazard Functions. Journal of Economic Literature, Vol. XXVI, pp. 646-679, Department of Economics, Stanford University, Stanford, USA.

LEE, L. (1980), Testing Adequacy of the Weibull and Log Lineair Rate Models for a Poisson Process. Technometrics, Vol. 22, No 2, pp. 195-199, The American Society for Quality Control and the American Statistical Association, Washington, D.C., USA.

LEE, L. (1980), Compairing Rates of Several Independent Weibull processes. Technometrics, Vol. 22, No 3, pp. 427-430, The American Society for Quality Control and the American Statistica Association, Washington, D.C., USA.

MALTHUS, T.R. (1798), Essay on Population. First publication, GB.

MEADOWS, D.L. (1972), The Limits to Growth: A Report for the Club of Rome project on the Predicament of Mankind. Universe Books, New York. A formal description of the original world model is contained in: FORRESTER, J.W. World Dynamics, 1971, Wright-Allen-Press, Cambridge, Massachussetts, USA.

MILL, J.S. (1857), Principles of Political Economy, dissertations and discussions. Vol. II, pp. 320-326, John W. Parker and Son, London, GB.

PEARSON, K. (1890-s), Frequence Curves and Method of Determining the Constants: in W. PALIN ELDERTON, Frequence Curves and Correlation. Charles and Edwin Layton, second edition, 1927, London, GB.

PEN, J. (1971), Income Distribution. ISBN 0-7139-0179-9, Allen Lane the Penguin Press, London, GB.

PICK, H.J. (1989), The Build-Up of Value in Manufacturing and Construction and in the Economy. Engineering Costs and Production Economics, No. 16, pp. 49-68, Elsevier Science Publishers, Amsterdam, Netherlands.

RAVENIS, J.V.J. (1964), Estimating Weibull-Distribution Parameters. Electro-Technology, pp. 46-54, Beverly Shores, USA.

RICARDO, D. (1817), Principles of Political Economy and Taxations. First Publication, GB.

ROWLEY, J.C.R. and TRIVEDI, P.K. (1973), Econometrics of Investment, John Wiley and Sons, New York, USA.

SEN, I. and PRABHASHANKER, V. (1980), A Nomogram for Estimating the Three Parameters of the Weibull Distribution. Journal of Quality Technology, Vol. 12, No. 3, pp. 138-143, American Society for Quality Control, Milwaukee, USA.

STEINECKE, V. (1975), Das Lebensdauernetz-Erläuterungen und Handhabung; Wahrscheinlichkeitspapier für die WEIBULL-Verteilung nach Stange und Gumbel. Deutsche Gesellschaft für Qualität, DGQ 25, Frankfurt am Main, FRG.

SVENSSON, L.G. and WEIBULL, J.W. (1986), An Upper Bound on Optimal Income Taxes. Journal of Public Economics, Vol. 30, No. 2, pp. 165-182, Amsterdam, Netherlands.

THUROW, L.C. (1980), The Zero-sum Society. Distribution and Possibilities for Economic Change. ISBN 0-465-09384-1, Basic Books, Inc., New York, USA.

U.S. TREASURY DEPARTMENT (1964), Internal Revenue Service. Depreciation, Guidelines and Rules, Washington, D.C., USA.

WARTNA, J.A. (1974), Bouw en Gebruik van Econometrische Modellen. ISBN 90-237-2064-4, University Press Rotterdam, Rotterdam, Netherlands.

WEMELSFELDER, J. (1988), Een Eenvoudige Theorie van "het Uitvinden" en enige Oefeningen in Semantiek. (Simple Theory of Inventions and some Experience in Semantics). ISBN 90-6778-013-8, Research Centre for Engineering Sciences, Innovation and Society, TWIM-Studies, Eindhoven, Netherlands.

APPENDICES VII

CODE	KIND OF EQUIPMENT	Purchase Price Class [Dutch Guilders (1975)]	Number of Discards 1976-1979
M.1.1.	Milling	10,000 - 40,000	83
M.2.1.	Lathes	10,000 - 35,000	62
M.2.2.	Lathes	more than 35,000	63
M.3.1.	Drilling	3,000 - 10,000	38
M.4.1.	Grinding	5,000 - 50,000	24
M.5.1.	Welding	unknown	17
M.6.1.	Surface treating	unknown	15
M.7.1.	Squeezing, punching and drawing	unknown	11

Table IV-1: 8 Sets of mechanically operated tools classified by different functions and discarded from one and the same engineering works in the period 1976-1979. Source: CBS (1982).

CODE	KIND OF EQUIPMENT	Purchase Price Class [Dutch Guilders (1980)]	Number of Discards 1980-1986
D. 1.1.	Passenger and delivery cars	more than 10,000	73
D. 2.1.	Lorries and trucks	more than 50,000	105
D. 3.1.	Internal transportation	10,000 - 50,000	168
D. 3.2.	Internal transportation	more than 50,000	67
D. 4.1.	Computers	?	54
D. 5.1.	Wrapping equipment	less than 10,000	49
D. 5.2.	Wrapping equipment	10,000 - 35,000	64
D. 5.3.	Wrapping equipment	more than 35,000	96
D. 6.1.	Pumps and compressors	10,000 - 20,000	162
D. 6.2.	Pumps and compressors	20,000 - 30,000	36
D. 6.3.	Pumps and compressors	more than 30,000	44
D. 7.1.	Electric generators	less than 10,000	150
D. 7.2.	Electric generators	more than 10,000	57
D. 8.1.	Welding and flame cutting	?	38
D. 9.1.	Measuring and controlling	less than 10,000	587
D. 9.2.	Measuring and controlling	more than 10,000	174
D.10.1.	Machining (chipping)	less than 12,500	118
D.10.2.	Machining (chipping)	12,500 - 55,000	43
D.11.1.	Machining (non-chipping)	less than 12,500	121
D.11.2.	Machining (non-chipping)	12,500 - 55,000	69

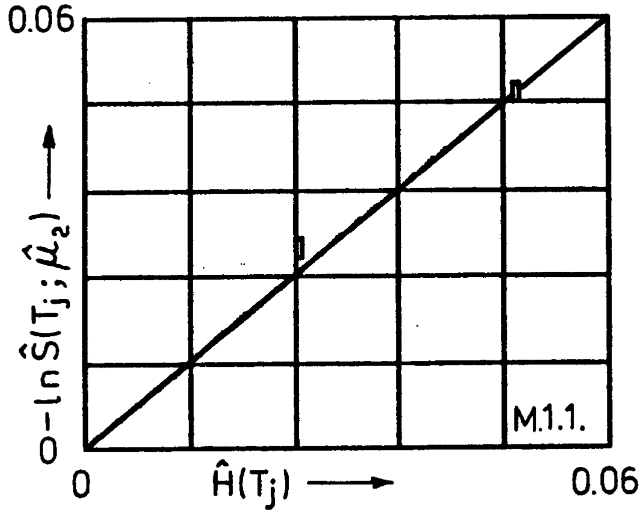
Table IV-2: 20 Sets of industrial capital equipment classified by different functions as recorded in Dalcy (database lifetime cycle). Source: CBS (1987).

CODE	KIND OF CAPITAL ASSETS	Retirement Period	Number of Discards
D.NL.12	Dwellings NL (12 points)	1961 - 1976	119,362
D.NL.48	Dwellings NL (48 points)	1961 - 1976	119,362
P.C.NL	Passenger cars NL	1977	≈300,000
B.T.NL	Bus tyres (The Hague, NL)	1981 - 1984	264

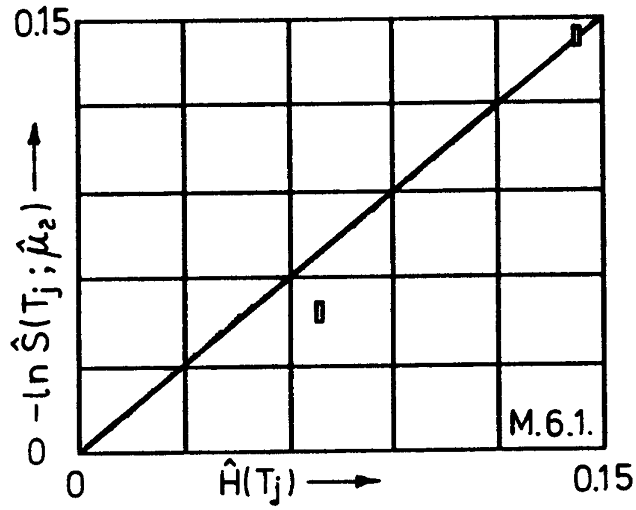
Table IV-4: Miscellaneous sets of capital assets. Sources: BEKKER (1980), VOORDOUW (1981) and TARIGAN (1985).

CODE	KIND OF PROPERTY	Retirement Period	Historical Value [$]	Number of Discards
1-1	Water works sources	-	-	18
2-1	Water works stations	-	-	23
3-1	Water works pumps	-	-	50
4-1	Water works steam engines	-	-	17
5-1	Water works boilers	-	-	32
6-1	Central office equipment (telephone)	-	1,238,925	-
7-1	Loading coils (telephone)	to 1917	265,835	-
8-1	Wooden telephone poles	-	-	2,423
9-1	Central office equipment (telephone)	-	848,109	-
10-1	Aerial cable (telephone)	to 1916	323,890	-
11-2	Aerial cable (telephone)	to 1915	386,910	-
12-1	Submarine cable (telephone)	to 1914	330,332	-
13-2	Submarine cable (telephone)	to 1914	87,296	-
14-1	Underground cable (telephone)	to 1914	1,433,383	-
15-2	Underground cable (telephone)	to 1915	759,329	-
16-3	Underground cable (telephone)	to 1916	457,251	-
17-4	Underground cable (telephone)	to 1914	9,526	-
18-5	Underground cable (telephone)	to 1915	31,850	-
19-6	Underground cable (telephone)	to 1916	53,101	-
20-1	Untreated wooden poles (telegraph)	-	5,708	-
21-2	Treated wooden poles (telegraph)	to 1904	-	124,300
22-3	Treated wooden poles (telegraph)	to 1904	-	77,606
23-4	Treated wooden poles (telegraph)	to 1904	-	11,084
24-5	Treated wooden poles (telegraph)	to 1904	-	30,009
25-1	Wooden poles (electricity supply)	1910 - 1915	-	1,372
26-2	Wooden poles (electricity supply)	-	-	309
27-1	Electric lamps (80W)	-	-	-
28-2	Electric lamps	-	-	-
29-3	Electric lamps (40W)	-	-	75
30-4	Mazda B-lamps (60W, electric)	-	-	100
31-1	Car wheel (electric railway)	1910	-	939
32-1	Railway stations	-	-	17
33-1	Steam locomotives (rail road)	-	-	781
34-1	Passenger cars (rail road)	-	-	>thousand
35-1	Freight cars (rail road)	-	-	15,372
36-1	Box cars (rail road)	-	-	8,788
37-1	Stock cars (rail road)	-	-	3,351
38-1	Flat cars (rail road)	to 1909	-	2,712
39-1	Wooden crossties (rail road)	-	-	9,937
40-2	Wooden crossties (rail road)	-	-	13,835
41-3	Wooden crossties (rail road)	-	-	1,000
42-4	Wooden crossties (rail road)	-	-	2,001
43-5	Wooden crossties (rail road)	-	-	26,146
44-6	Wooden crossties (rail road)	-	-	43,681
45-7	Wooden crossties (rail road)	-	-	12,951
46-8	Wooden crossties (rail road)	-	-	20,536
47-9	Wooden crossties (rail road)	-	-	4,786
48-10	Wooden crossties (rail road)	-	-	1,048
49-11	Wooden crossties (rail road)	-	-	1,000
50-12	Wooden crossties (rail road)	-	-	5,909
51-13	Wooden crossties (rail road)	-	-	137,000
52-14	Wooden crossties (rail road)	-	-	2,916
53-1	Rodger ballast cars (rail road)	-	-	760
54-2	Box cars (rail road)	1869 - 1923	-	1,107
55-2	Flat cars (rail road)	1869 - 1923	-	3,114
56-1	Corn cultivators (1-row)	1875 - 1924	-	56
57-1	Corn planters	1874 - 1924	-	55
58-1	Disc harrows (8-foot)	1875 - 1924	-	43
59-1	Grain binders (5 to 8-foot)	1882 - 1924	-	45
60-1	Manure spreaders	1890 - 1924	-	37
61-1	Mowers	1878 - 1924	-	37
62-1	Plows	1880 - 1924	-	30
63-1	Passenger automobiles	1911 & 1913	-	unknown
64-2	Passenger automobiles	1922	-	3,124
65-3	Passenger automobiles	1926	-	9,878

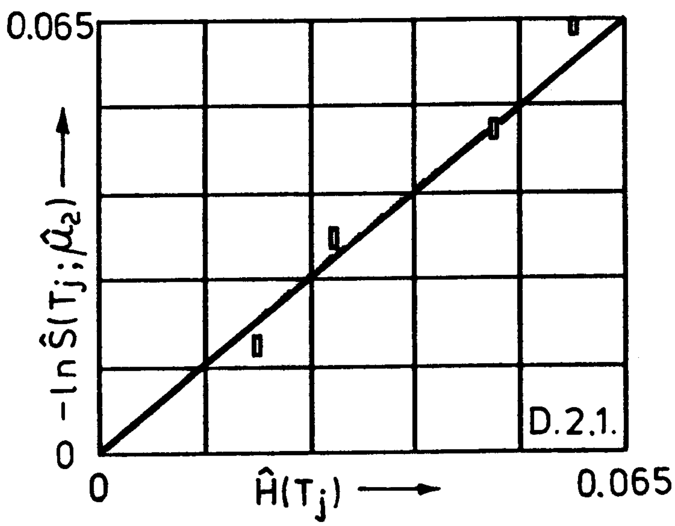
Table IV-3: 65 Sets of physical property. Source: IOWA Bulletin 103, WINFREY (1931).



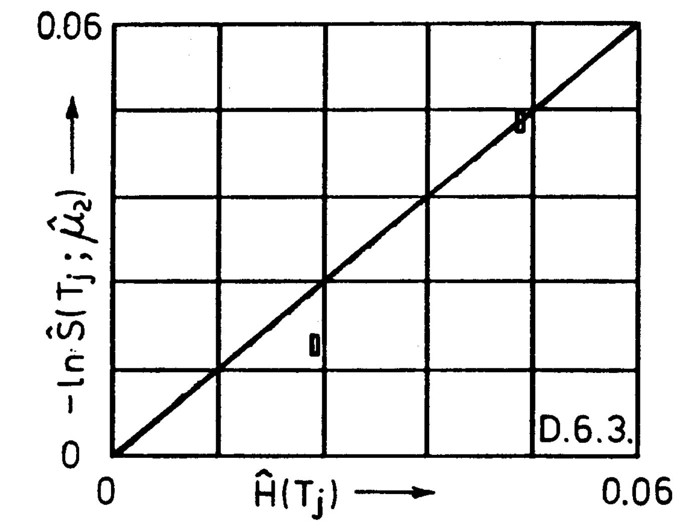
H-residuals Phase II:
Milling equipment,
Code M.1.1.



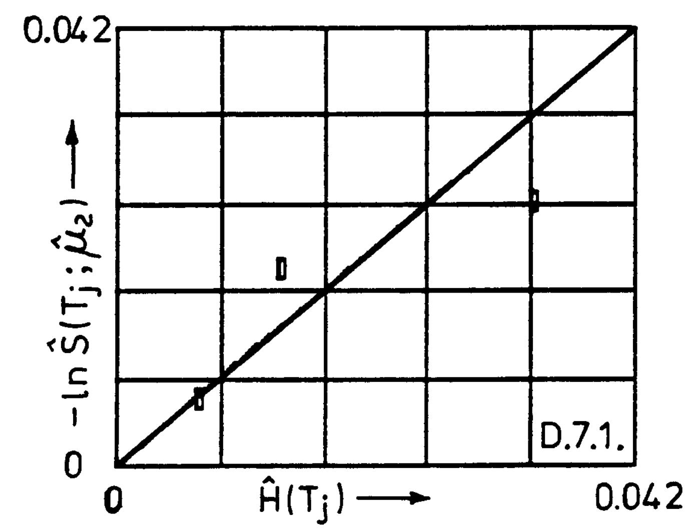
H-residuals Phase II:
Surface-treating equipment,
Code M.6.1.



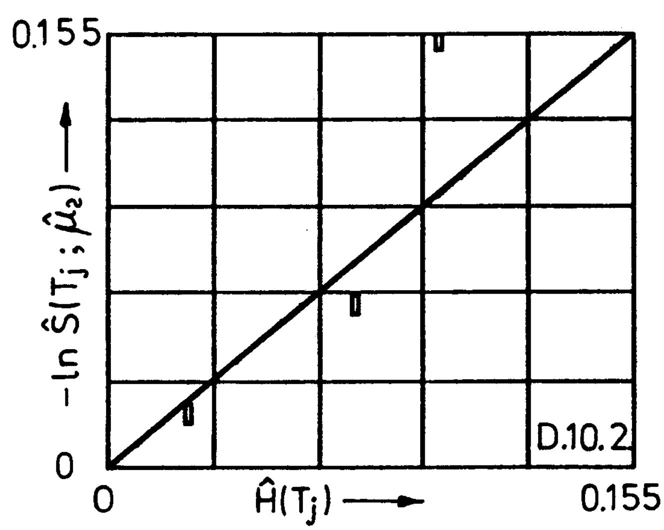
H-residuals Phase II:
Lorries and trucks,
Code D.2.1.



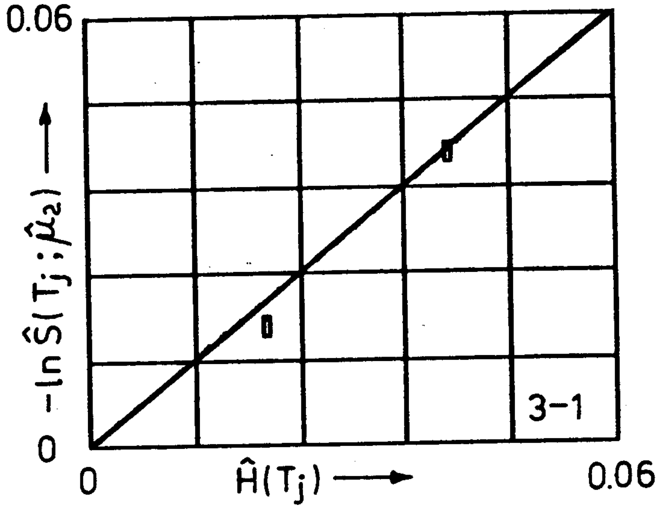
H-residuals Phase II:
Pumps and Compressors,
Code D.6.3.



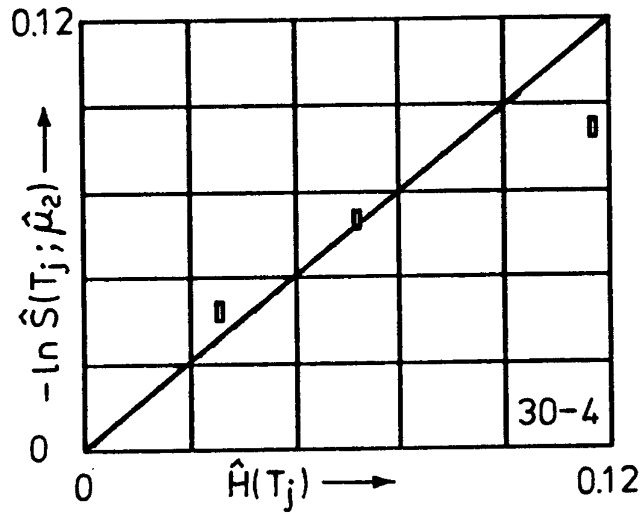
H-residuals Phase II:
Electric generators,
Code D.7.1.



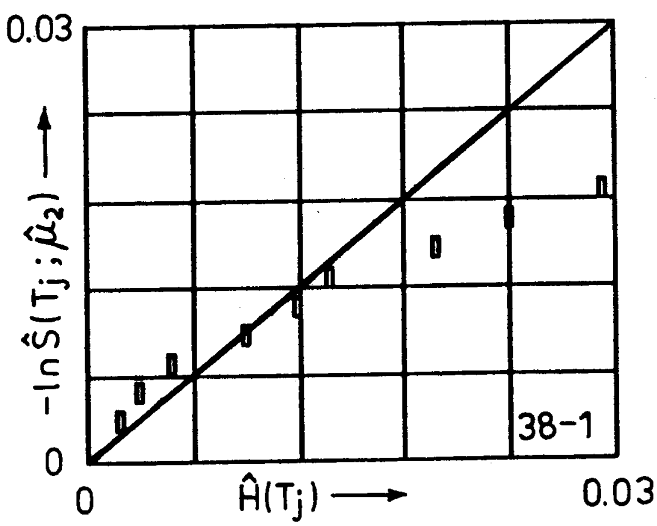
H-residuals Phase II:
Machining equipment,
Code D.10.2



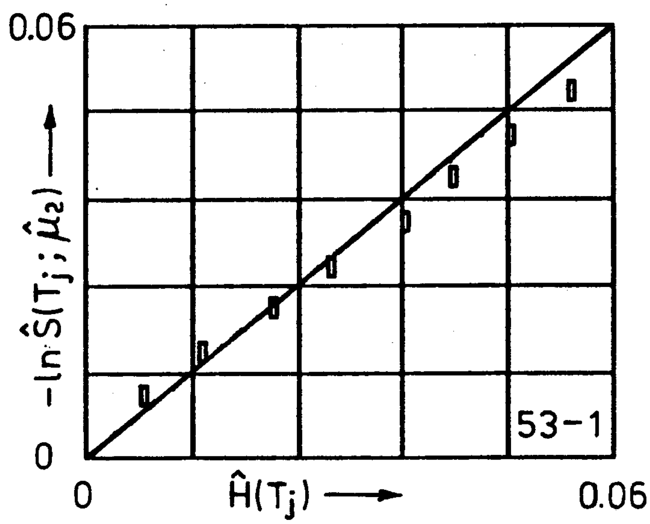
H-residuals Phase II:
Water works pumps,
Code 3-1.



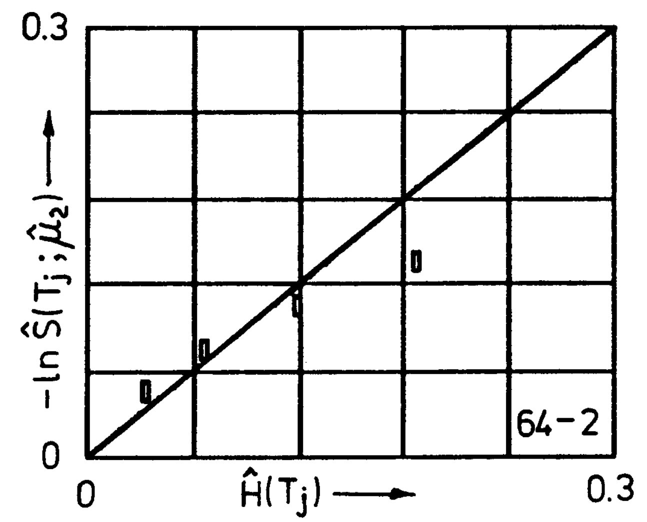
H-residuals Phase II:
Mazda B-lamps (60W),
Code 30-4.



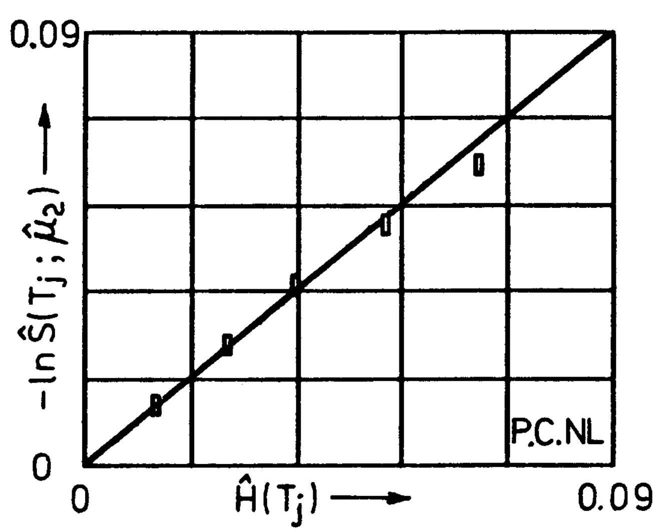
H-residuals Phase II:
Coal flat train cars,
Code 38-1.



H-residuals Phase II:
Rodger ballast train cars,
Code 53-1.



H-residuals Phase II:
Automobiles 1900-1922,
Code 64-2.



H-residuals Phase II:
Passenger cars,
Code P.C.NL.

CODE	KIND OF EQUIPMENT	Estimated Parameters			
		$\hat{\beta}_3$	$\hat{\mu}_3$	(a)	r
M.1.1.	Milling	4.83	13.32	6.78	0.998
M.2.1.	Lathes	5.98	13.56	8.03	0.994
M.2.2.	Lathes	5.53	14.63	8.09	0.960
M.3.1.	Drilling	6.47	15.62	9.45	0.977
M.4.1.	Grinding	3.29	13.93	4.41	0.985
M.5.1.	Welding	3.47	14.91	5.00	0.978
M.6.1.	Surface treating	2.90	16.68	3.78	0.992
M.7.1.	Squeezing, punching and drawing	3.13	13.60	3.98	0.989

Table IV-6: Preliminary WEIBULL parameter estimates concerning mechanically operated tools listed in Table IV-1, Appendix IX.1., page 1.

CODE	KIND OF EQUIPMENT	Estimated Parameters			
		$\hat{\beta}_3$	$\hat{\mu}_3$	(a)	r
D. 1.1.	Passenger and delivery cars	2.38	6.72	1.69	0.995
D. 2.1.	Lorries and trucks	3.26	10.20	3.65	0.992
D. 3.1.	Internal transportation	2.27	13.35	1.73	0.997
D. 3.2.	Internal transportation	2.09	13.81	1.24	0.993
D. 4.1.	Computers	1.76	6.20	0.57	0.977
D. 5.1.	Wrapping equipment	1.61	13.73	0.18	0.991
D. 5.2.	Wrapping equipment	1.78	13.18	0.48	0.997
D. 5.3.	Wrapping equipment	2.16	14.66	1.44	0.966
D. 6.1.	Pumps and compressors	2.73	17.94	3.37	0.988
D. 6.2.	Pumps and compressors	2.78	15.80	3.36	0.979
D. 6.3.	Pumps and compressors	2.74	17.82	3.40	0.994
D. 7.1.	Electric generators	2.47	16.74	2.47	0.998
D. 7.2.	Electric generators	1.96	14.36	0.90	0.993
D. 8.1.	Welding and flame cutting	2.29	19.70	1.95	0.994
D. 9.1.	Measuring and controlling	2.02	16.25	1.05	0.999
D. 9.2.	Measuring and controlling	1.89	15.95	0.72	0.997
D.10.1.	Machining (chipping)	1.89	16.37	0.71	0.988
D.10.2.	Machining (chipping)	2.20	23.45	1.70	0.991
D.11.1.	Machining (non-chipping)	1.76	14.13	0.44	0.990
D.11.2.	Machining (non-chipping)	1.49	18.30	0.05	0.982

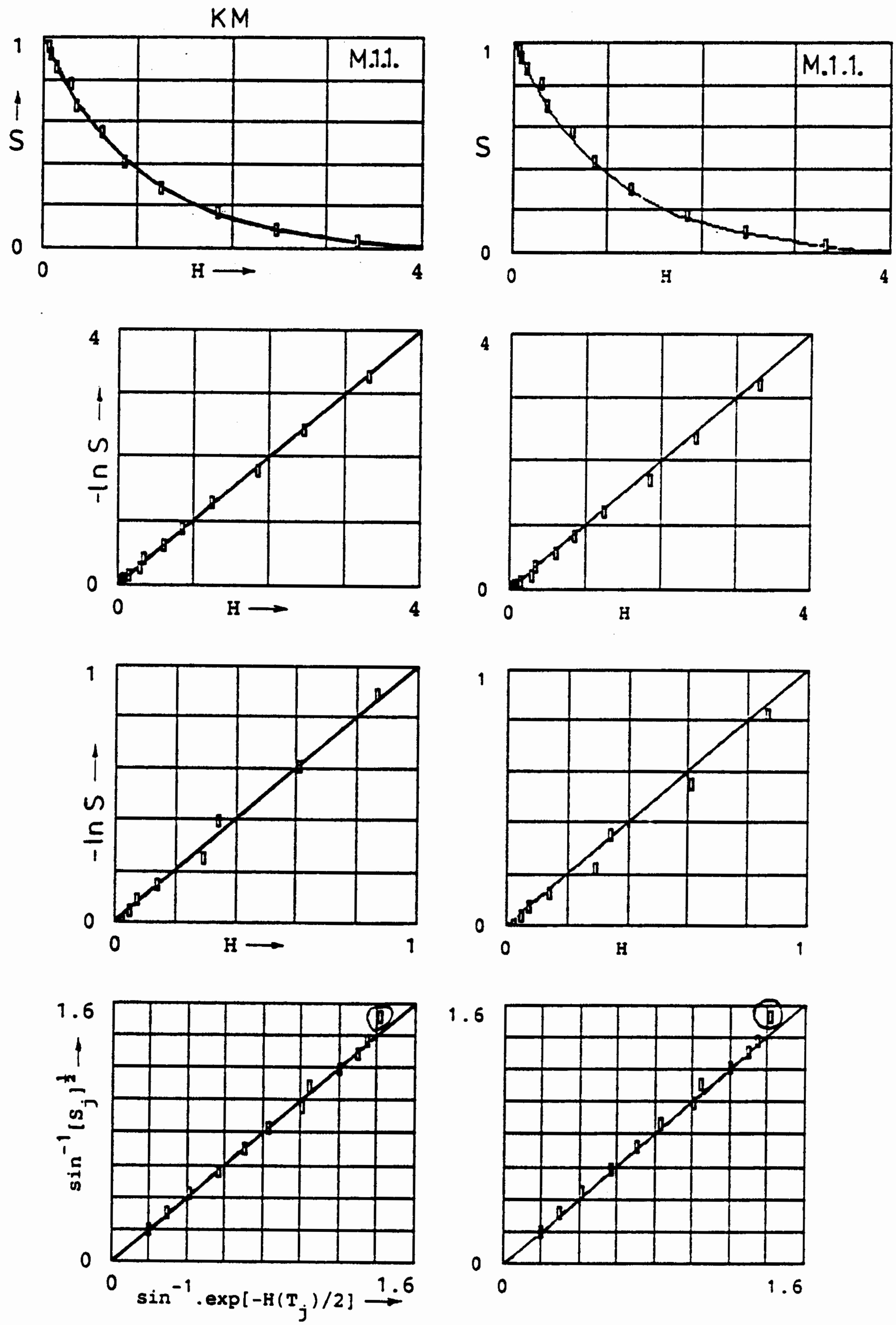
Table IV-7: Preliminary WEIBULL parameter estimates concerning Dalcy (database lifetime cycle) listed in Table IV-2, Appendix IX-1., page 1.

CODE	KIND OF CAPITAL ASSETS	Estimated Parameters			
		$\hat{\beta}_3$	$\hat{\mu}_3$	(a)	r
D.NL.12	Dwellings NL (12 points)	3.88	102.27	20.52	0.998
D.NL.48	Dwellings NL (48 points)	3.65	93.18	16.83	0.990
P.C.NL	Passenger cars NL	4.39	9.42	4.86	0.993
B.T.NL	Bus tyres (The Hague, NL)	3.28	144,208	16,304	0.998

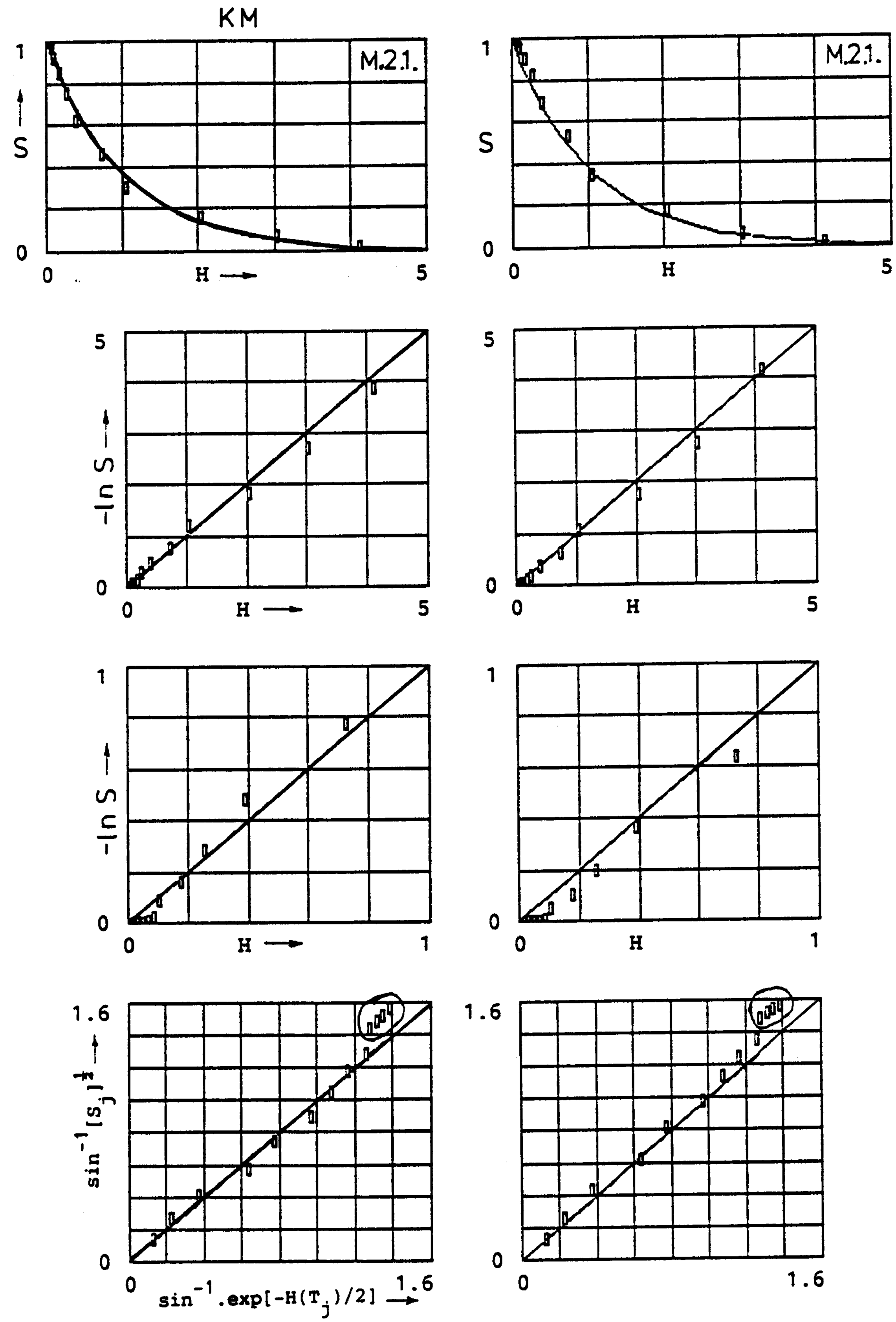
Table IV-9: Preliminary WEIBULL parameter estimates concerning miscellaneous sets of capital assets listed in Table IV-4, Appendix VII.1, page 1.

CODE	KIND OF CAPITAL ASSETS	Estimated Parameters			
		$\hat{\beta}_3$	$\hat{\mu}_3$	(a)	r
1-1	Water works sources	1.862	18.201	0.629	0.974
2-1	Water works stations	1.643	16.792	0.209	0.986
3-1	Water works pumps	2.646	23.585	3.459	0.988
4-1	Water works steam engines	3.847	33.717	9.801	0.984
5-1	Water works boilers	3.578	16.518	5.566	0.970
6-1	Central office equipment (telephone)	1.982	6.162	0.966	0.955
7-1	Loading coils (telephone)	6.422	12.862	8.030	0.986
8-1	Wooden telephone poles	2.323	14.015	1.905	0.983
9-1	Central office equipment (telephone)	1.659	9.767	0.308	0.995
10-1	Aerial cable (telephone)	1.986	12.084	0.966	0.999
11-2	Aerial cable (telephone)	2.575	10.902	2.392	1.000
12-1	Submarine cable (telephone)	1.595	10.784	0.199	0.983
13-2	Submarine cable (telephone)	2.102	13.181	1.269	0.989
14-1	Underground cable (telephone)	2.225	15.988	1.664	0.992
15-2	Underground cable (telephone)	2.364	17.507	2.148	0.995
16-3	Underground cable (telephone)	2.243	18.630	1.771	0.982
17-4	Underground cable (telephone)	2.144	13.587	1.388	0.992
18-5	Underground cable (telephone)	1.878	13.993	0.693	0.998
19-6	Underground cable (telephone)	1.621	13.356	0.205	0.991
20-1	Untreated wooden poles (telegraph)	3.076	10.155	3.324	0.973
21-2	Treated wooden poles (telegraph)	2.616	10.912	2.488	0.998
22-3	Treated wooden poles (telegraph)	2.325	10.821	1.793	0.982
23-4	Treated wooden poles (telegraph)	3.279	10.892	3.820	0.989
24-5	Treated wooden poles (telegraph)	2.945	11.720	3.306	0.998
25-1	Wooden poles (electricity supply)	2.149	11.315	1.369	0.993
26-2	Wooden poles (electricity supply)	2.737	13.128	2.982	0.993
27-1	Electric lamps (80W)	3.428	9.628	3.789	0.993
28-2	Electric lamps	2.361	11.696	1.921	0.997
29-3	Electric lamps (40W)	4.990	13.250	6.934	0.980
30-4	Mazda B-lamps (60W, electric)	3.763	6.387	3.263	0.995
31-1	Car wheel (electric railway)	2.230	6.899	1.434	0.997
32-1	Railway stations	2.094	26.089	1.325	0.974
33-1	Steam locomotives (rail road)	4.000	28.903	9.418	0.991
34-1	Passenger cars (rail road)	4.751	36.714	14.049	0.996
35-1	Freight cars (rail road)	4.302	21.369	8.455	0.987
36-1	Box cars (rail road)	5.267	22.586	10.879	0.998
37-1	Stock cars (rail road)	4.055	19.800	7.450	0.992
38-1	Flat cars (rail road)	4.168	21.073	8.051	0.993
39-1	Wooden crossties (rail road)	6.135	11.891	7.343	0.990
40-2	Wooden crossties (rail road)	4.620	10.096	5.331	0.979
41-3	Wooden crossties (rail road)	14.771	13.474	11.155	0.996
42-4	Wooden crossties (rail road)	10.514	18.170	13.396	0.978
43-5	Wooden crossties (rail road)	7.235	10.076	6.956	0.980
44-6	Wooden crossties (rail road)	6.990	11.749	7.787	0.988
45-7	Wooden crossties (rail road)	5.314	11.360	6.468	0.959
46-8	Wooden crossties (rail road)	5.674	12.112	7.103	0.991
47-9	Wooden crossties (rail road)	5.590	10.815	6.438	0.998
48-10	Wooden crossties (rail road)	7.291	11.003	7.516	0.946
49-11	Wooden crossties (rail road)	4.395	12.695	5.975	0.949
50-12	Wooden crossties (rail road)	2.877	11.426	3.121	0.896
51-13	Wooden crossties (rail road)	5.946	9.258	5.903	0.979
52-14	Wooden crossties (rail road)	5.810	8.554	5.475	1.000
53-1	Rodger ballast cars (rail road)	3.498	21.993	6.382	0.974
54-2	Box cars (rail road)	3.232	31.385	6.701	0.979
55-2	Flat cars (rail road)	2.683	28.978	3.918	0.995
56-1	Corn cultivators (1-row)	3.213	14.426	4.318	0.990
57-1	Corn planters	1.913	12.387	0.787	0.997
58-1	Disc harrows (8-foot)	1.983	14.814	0.956	0.989
59-1	Grain binders (5 to 8-foot)	2.550	16.338	2.695	0.988
60-1	Manure spreaders	2.395	11.667	2.004	0.997
61-1	Mowers	1.913	12.407	0.787	0.988
62-1	Plows	2.260	13.814	1.718	0.943
63-1	Passenger automobiles	1.737	6.213	0.522	0.993
64-2	Passenger automobiles	3.224	7.119	2.945	0.998
65-3	Passenger automobiles	3.150	8.808	3.202	0.996

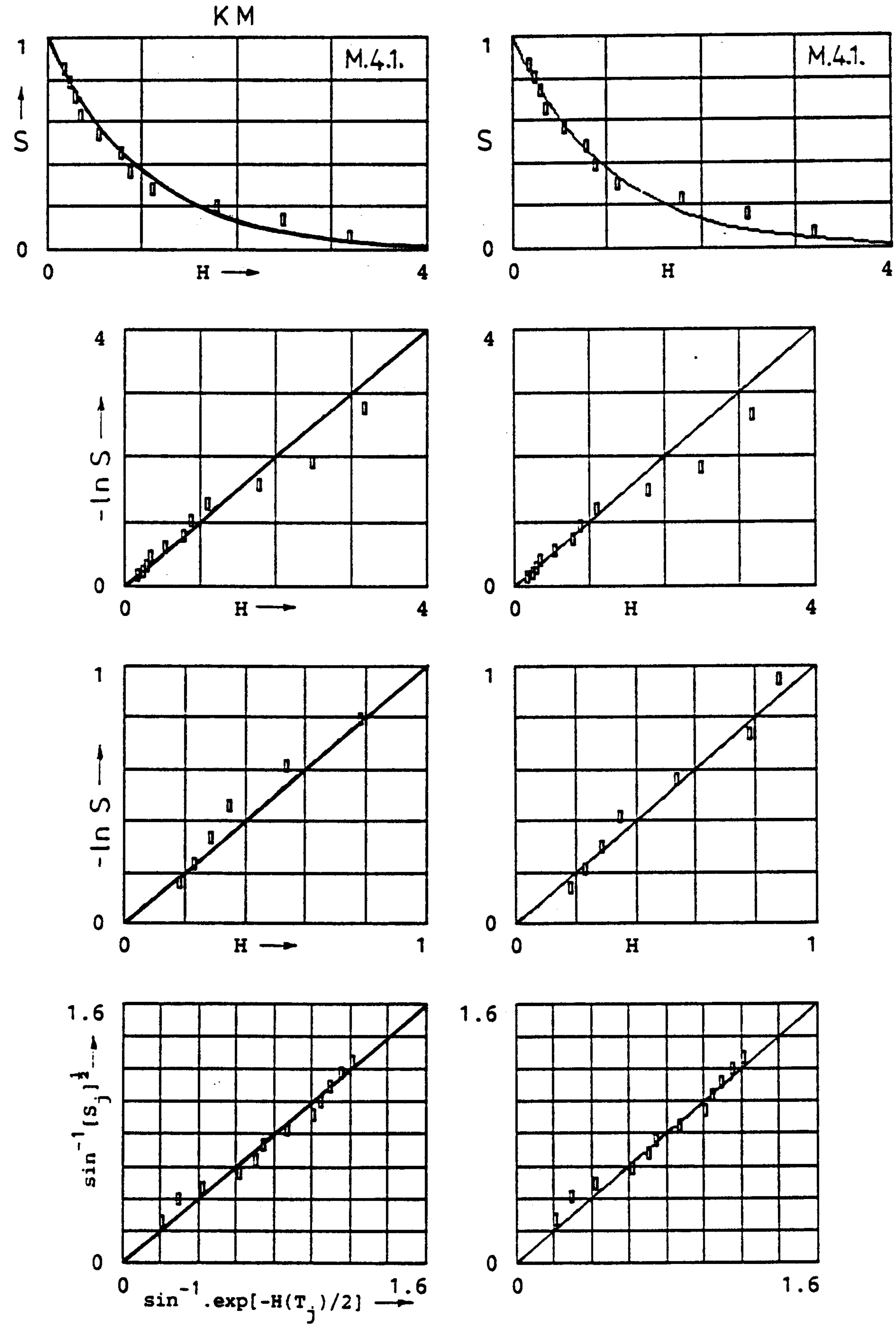
Table IV-8: Preliminary WEIBULL parameter estimates concerning WINFREY (1931) physical property listed in Table IV-3, Appendix VII.1, page 2.



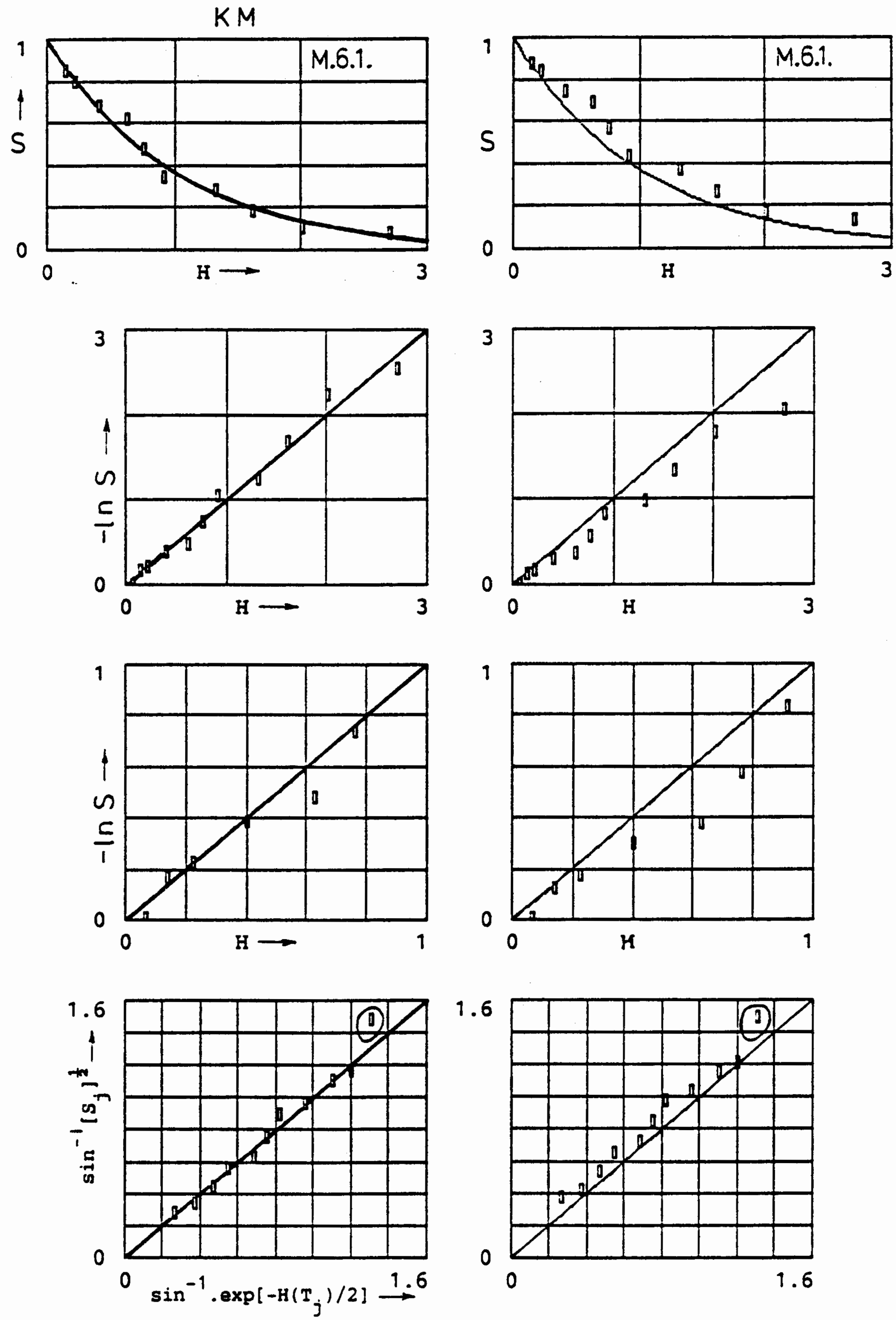
Plot of H-residuals: Milling Equipment, Code M.1.1.



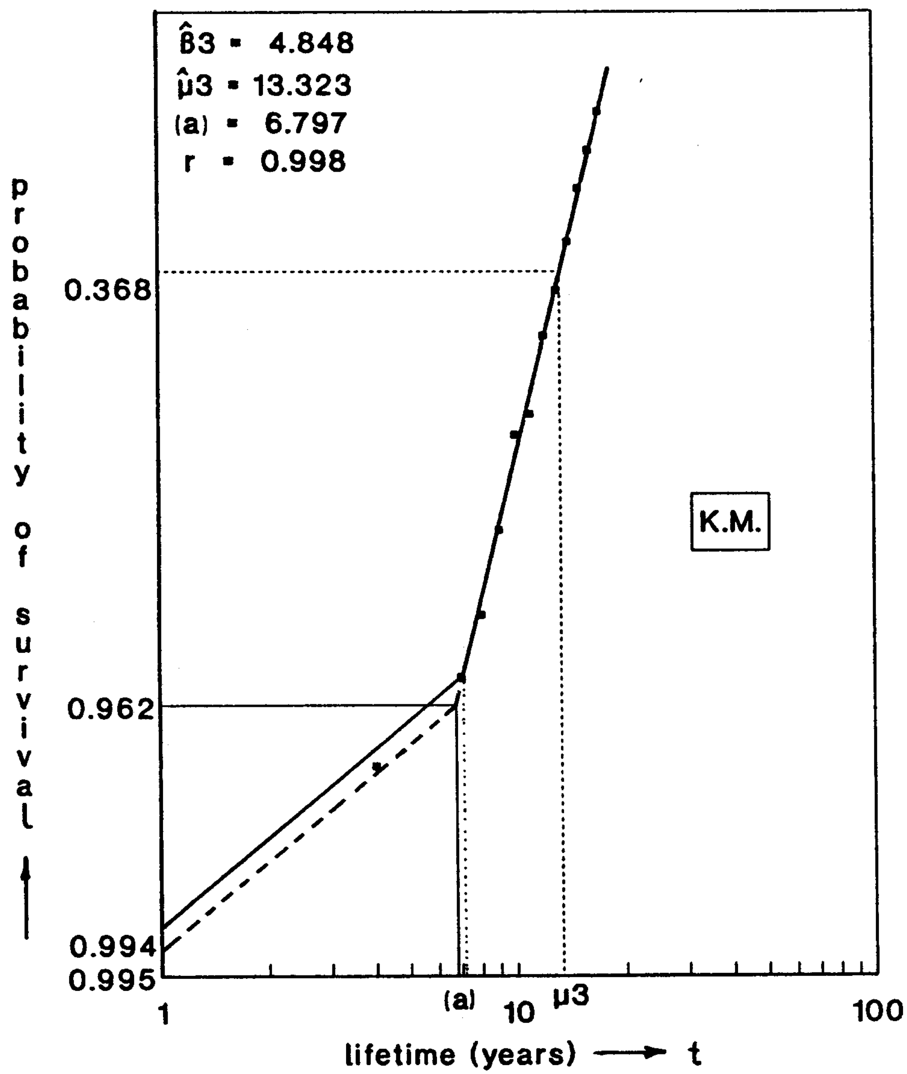
Plot of H-residuals: Lathes, Code M.2.1.



Plot of H-residuals: Grinding equipment, Code M.4.1.

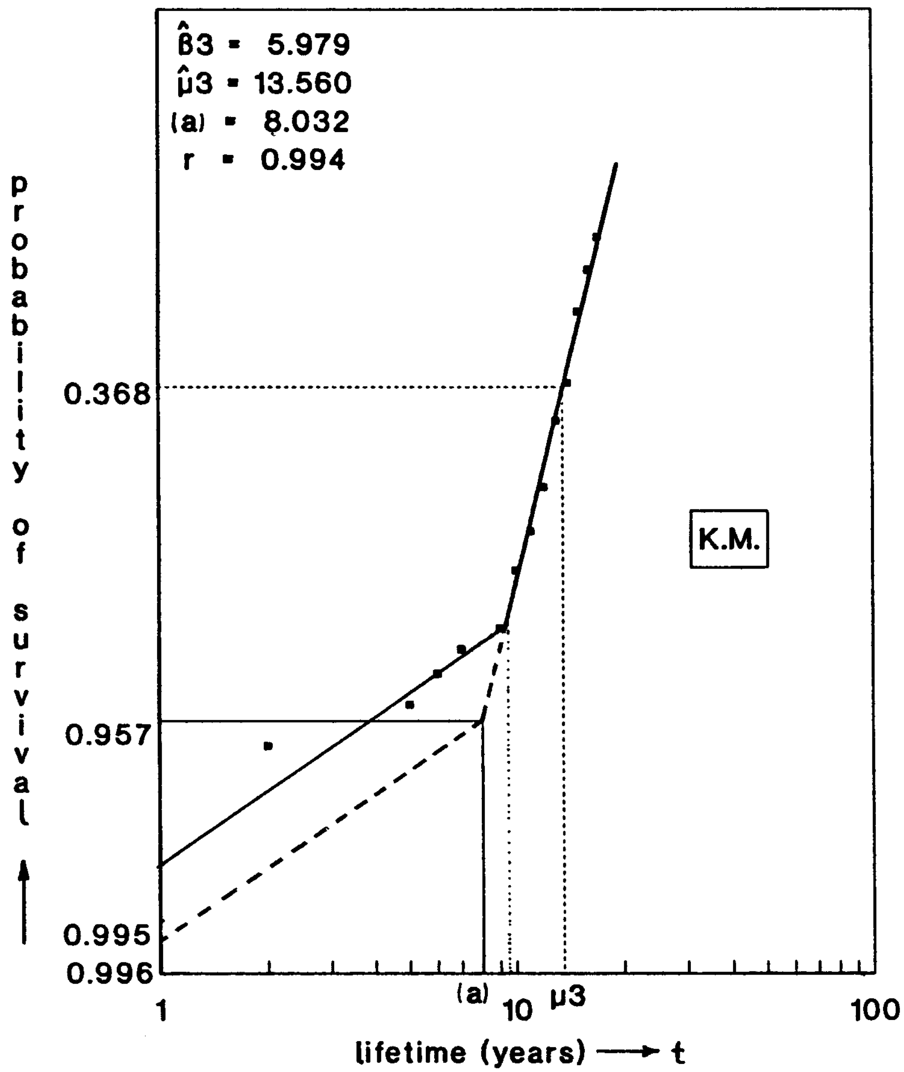


Plot of H-residuals: Surface-treating equipment, Code M.6.1.



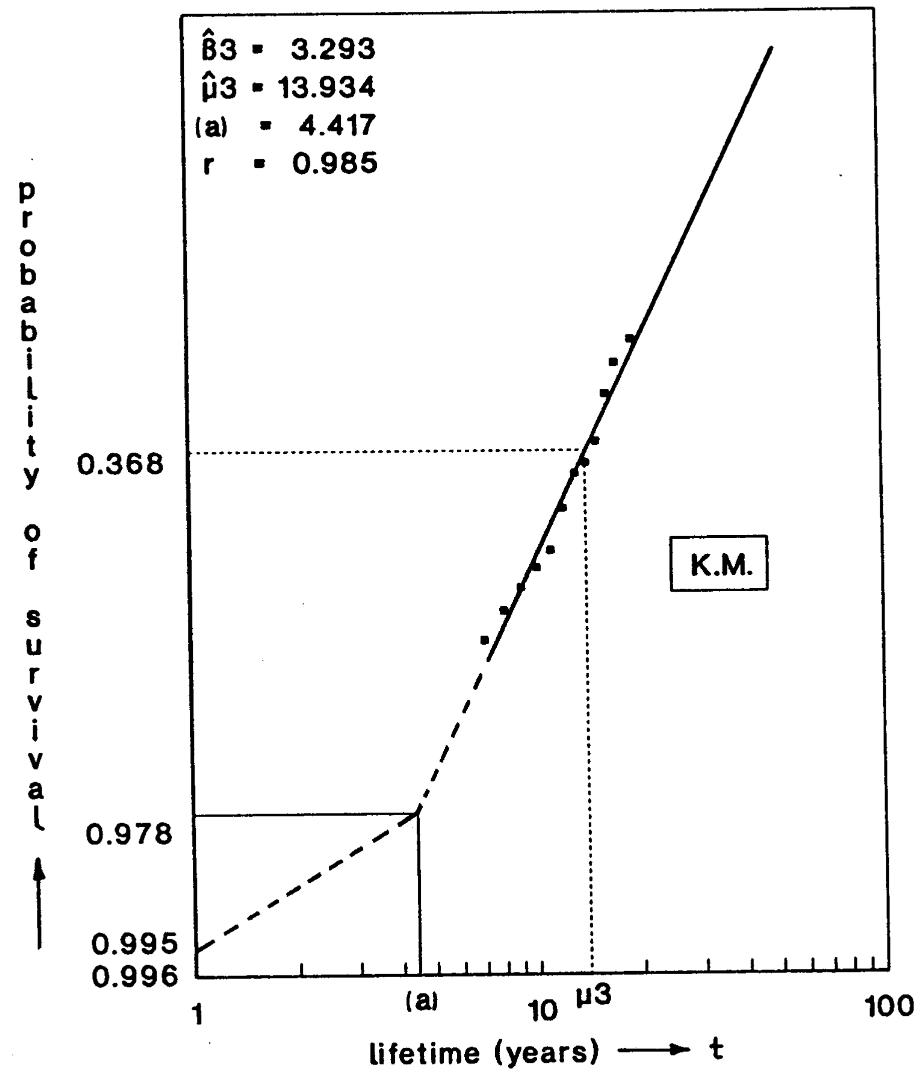
M 1.1 Milling equipment

T_j	d_j	$\hat{S}_j$
4	2	0.976
7	2	0.952
8	2	0.928
9	5	0.867
10	10	0.747
11	3	0.711
12	14	0.542
13	10	0.422
14	11	0.289
15	11	0.157
16	6	0.084
17	4	0.036
19	3	0.000



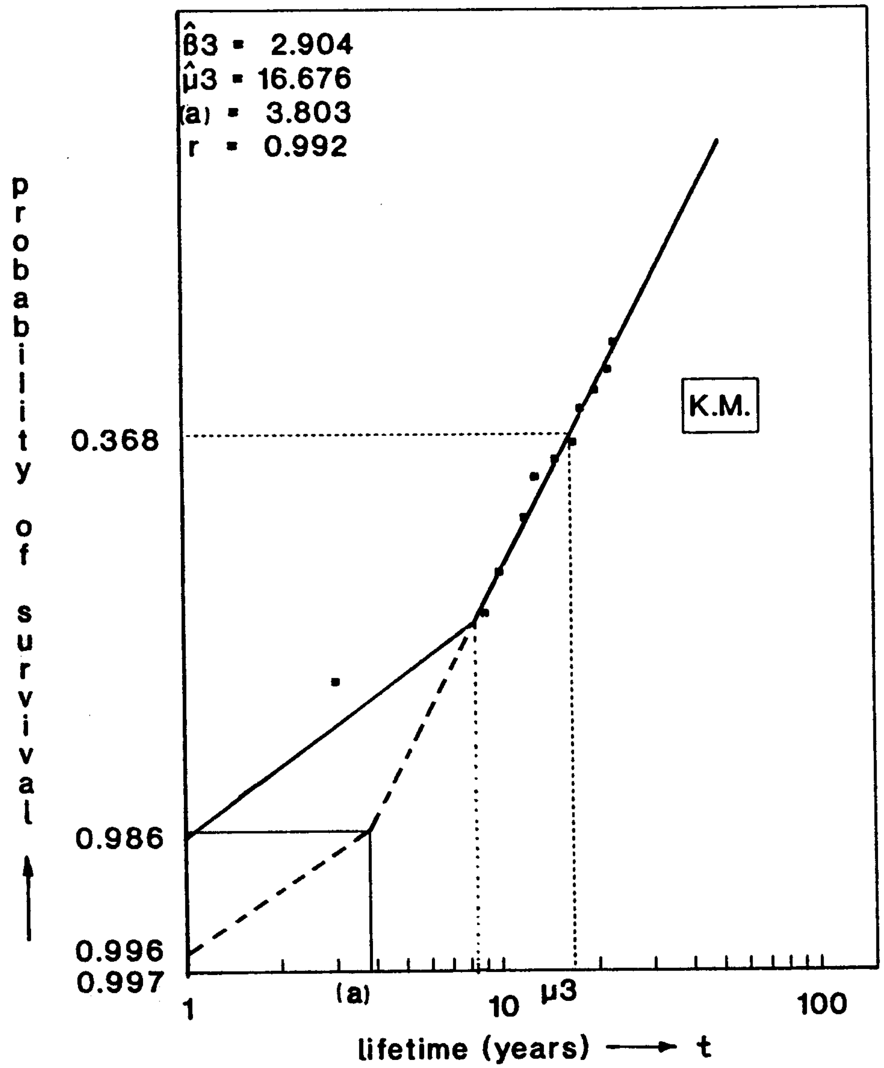
M 2.1 Lathes

τ_j	d_j	$\hat{S}_j$
2	2	0.968
5	1	0.952
6	1	0.935
7	1	0.919
9	1	0.903
10	4	0.839
11	4	0.774
12	6	0.677
13	12	0.484
14	8	0.355
15	14	0.129
16	5	0.048
17	2	0.016
18	1	0.000



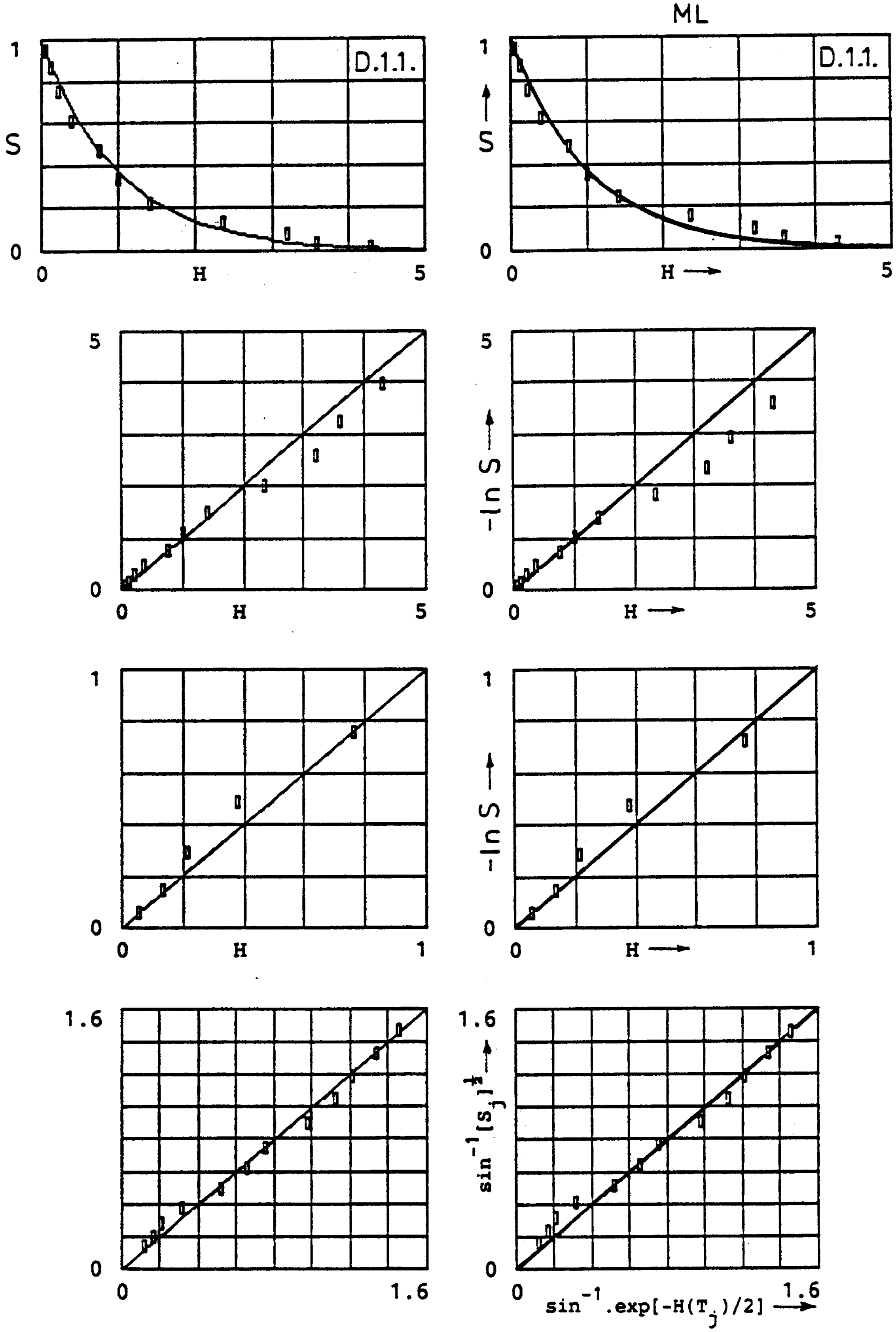
M 4.1 Grinding equipment

τ_j	d_j	$\hat{S}_j$
7	3	0.875
8	1	0.833
9	1	0.792
10	1	0.750
11	1	0.708
12	3	0.583
13	3	0.458
14	1	0.417
15	2	0.333
16	4	0.167
17	2	0.083
19	1	0.042
24	1	0.000

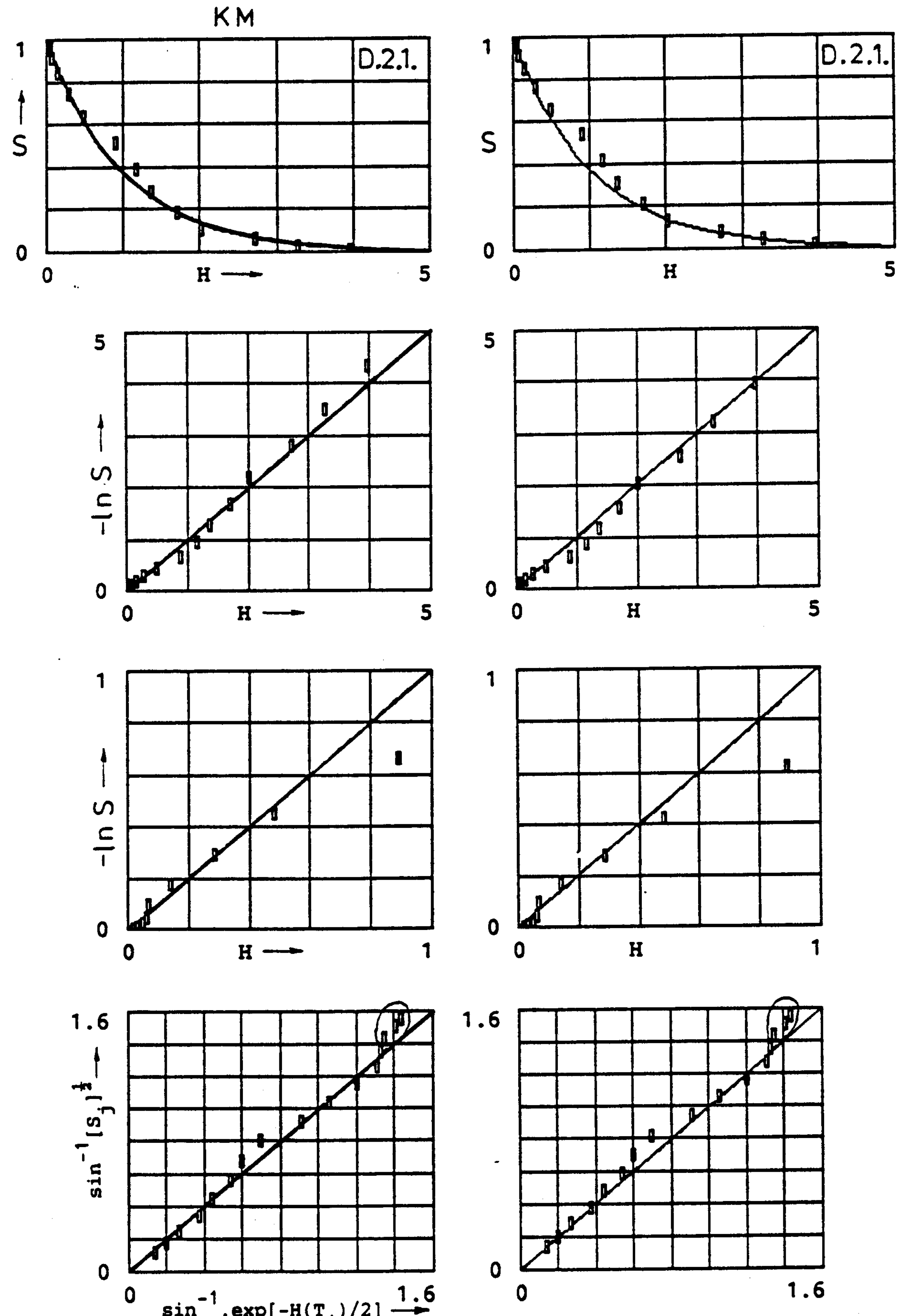


M 6.1 Surface-treating equipment

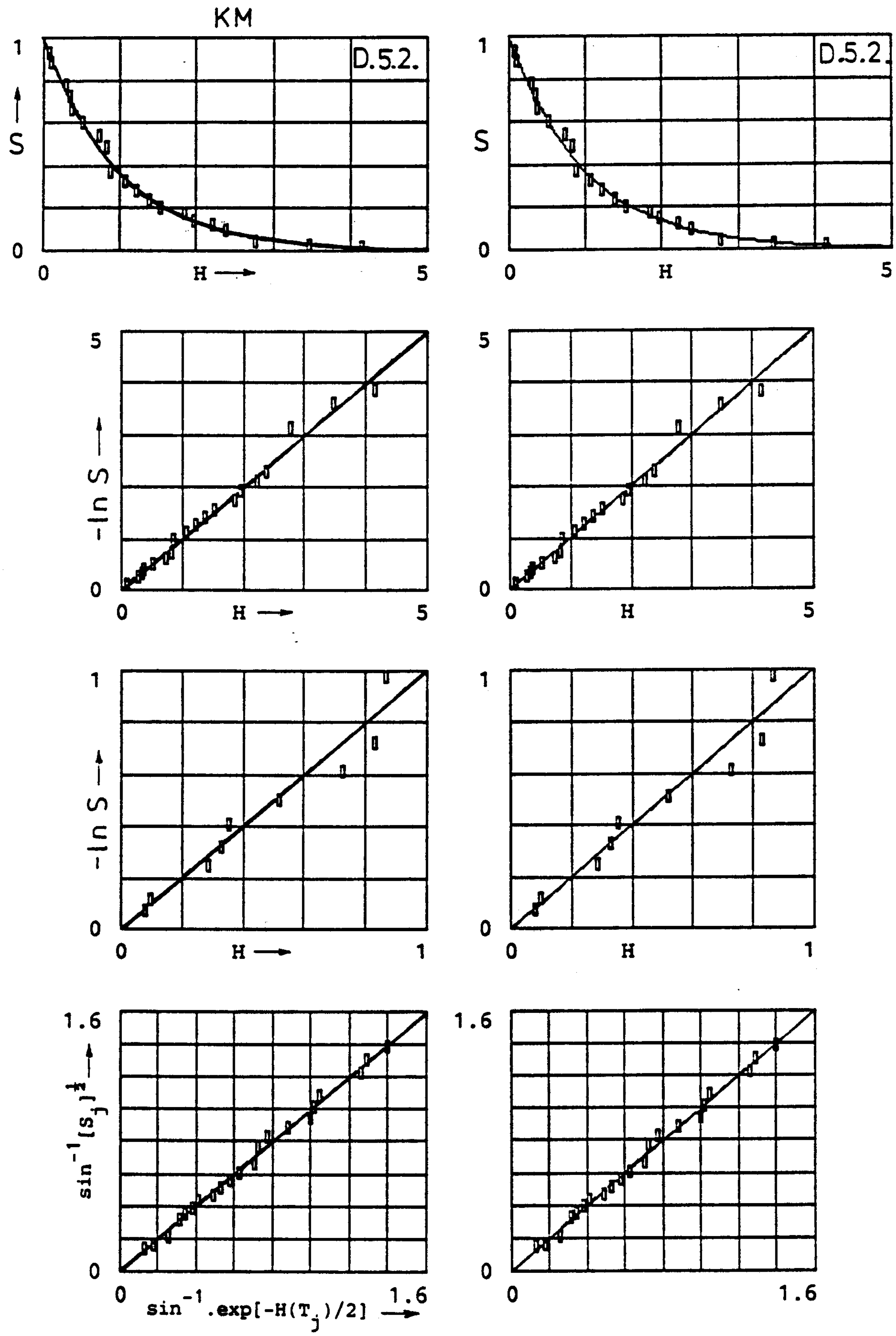
T_j	d_j	$\hat{S}_j$
3	1	0.933
9	1	0.867
10	1	0.800
12	2	0.667
13	2	0.533
15	1	0.467
17	1	0.400
18	2	0.267
20	1	0.200
22	1	0.133
23	1	0.067
31	1	0.000



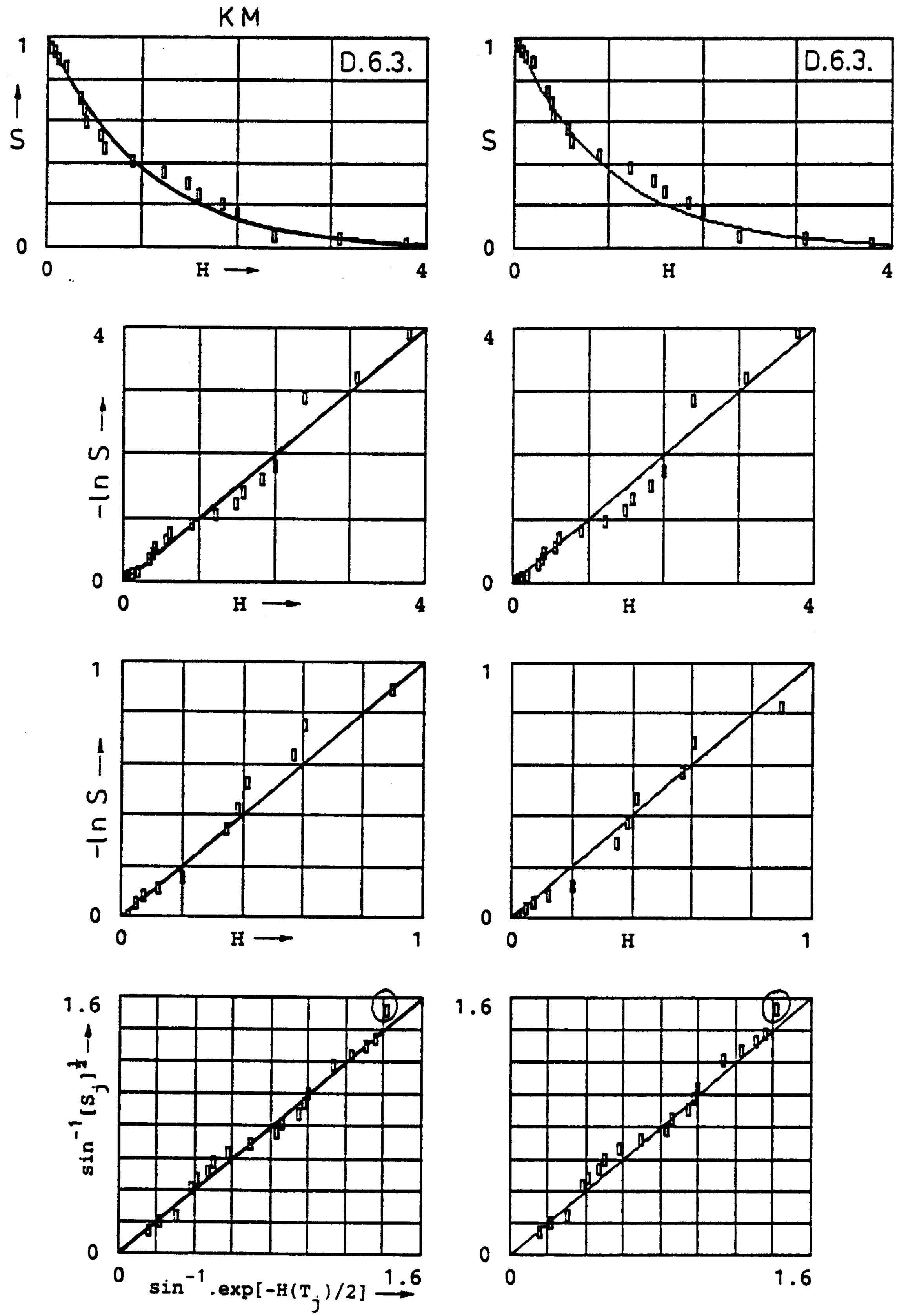
Plot of H-residuals: Passenger and delivery cars, Code D.1.1.



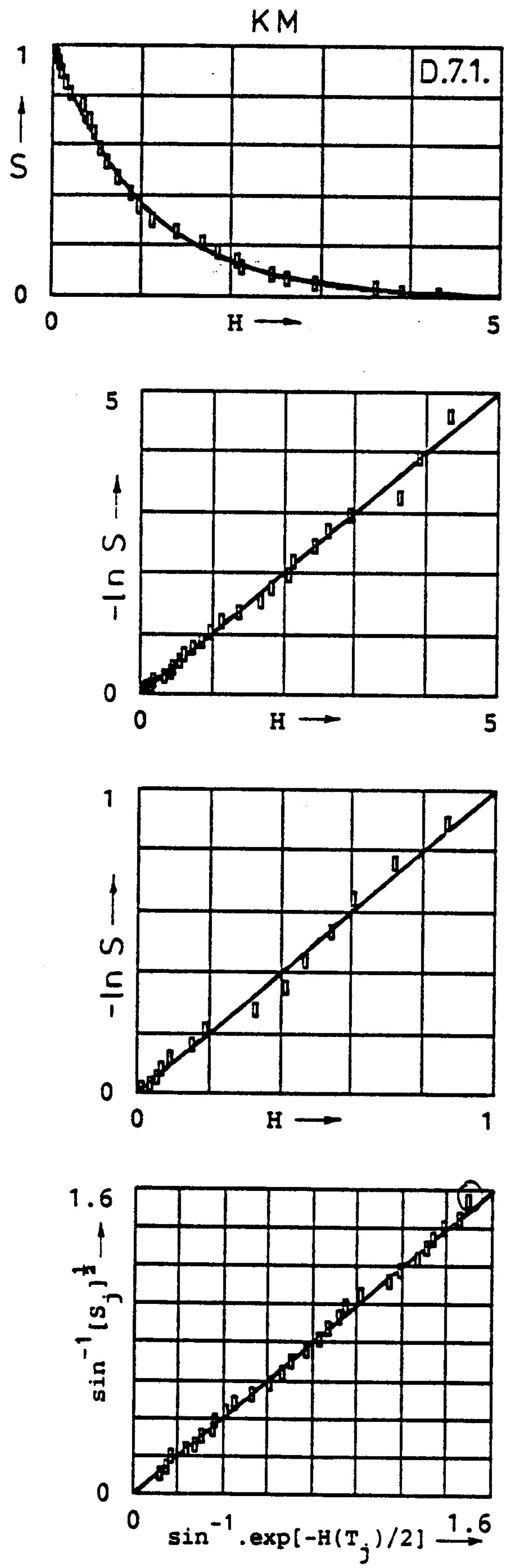
Plot of H-residuals: Lorries and trucks, Code D.2.1.



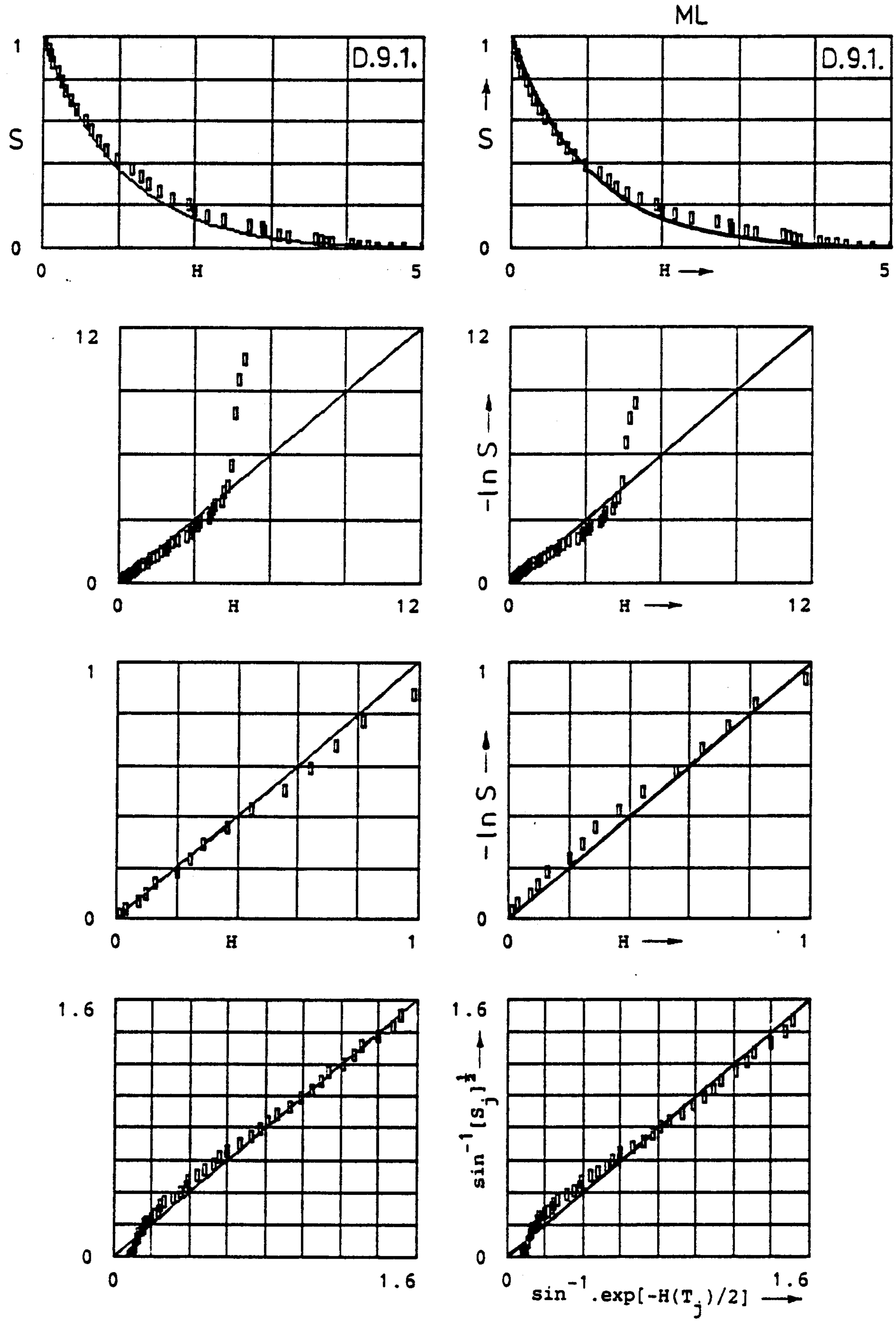
Plot of H-residuals: Wrapping equipment, Code D.5.2.



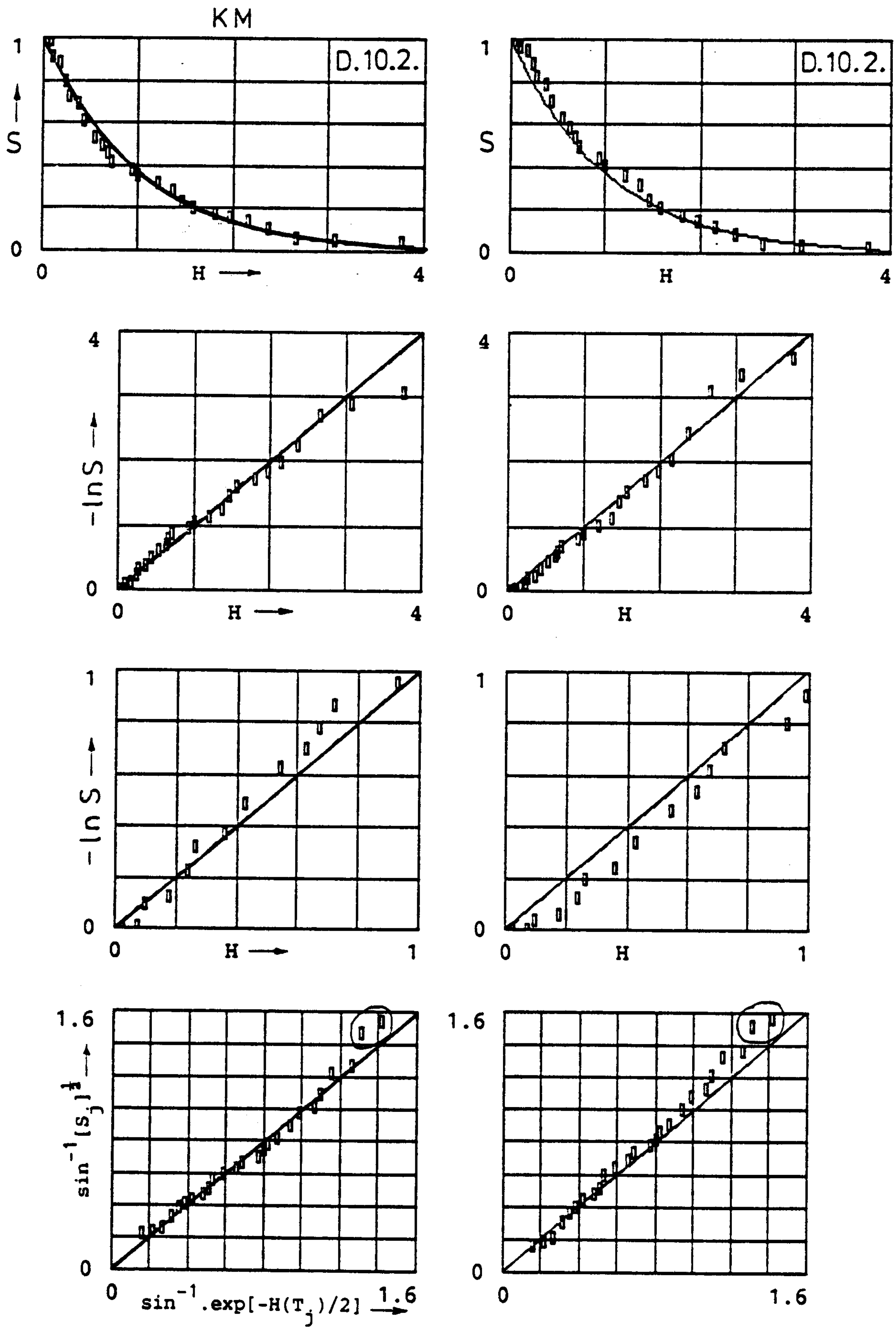
Plot of H-residuals: Pumps and compressors, Code D.6.3.



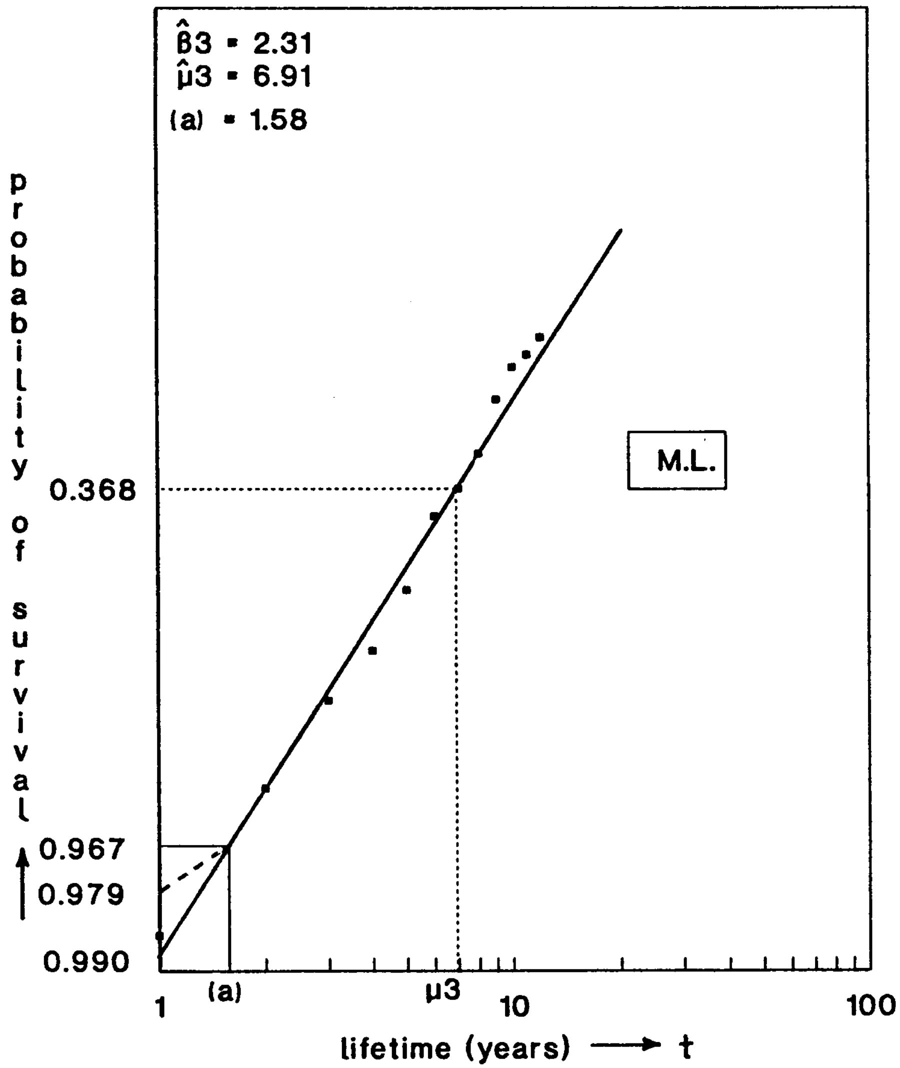
Plot of H-residuals: Electric generators, Code D.7.1.



Plot of H-residuals: Measuring and controlling equipment, Code D.9.1.

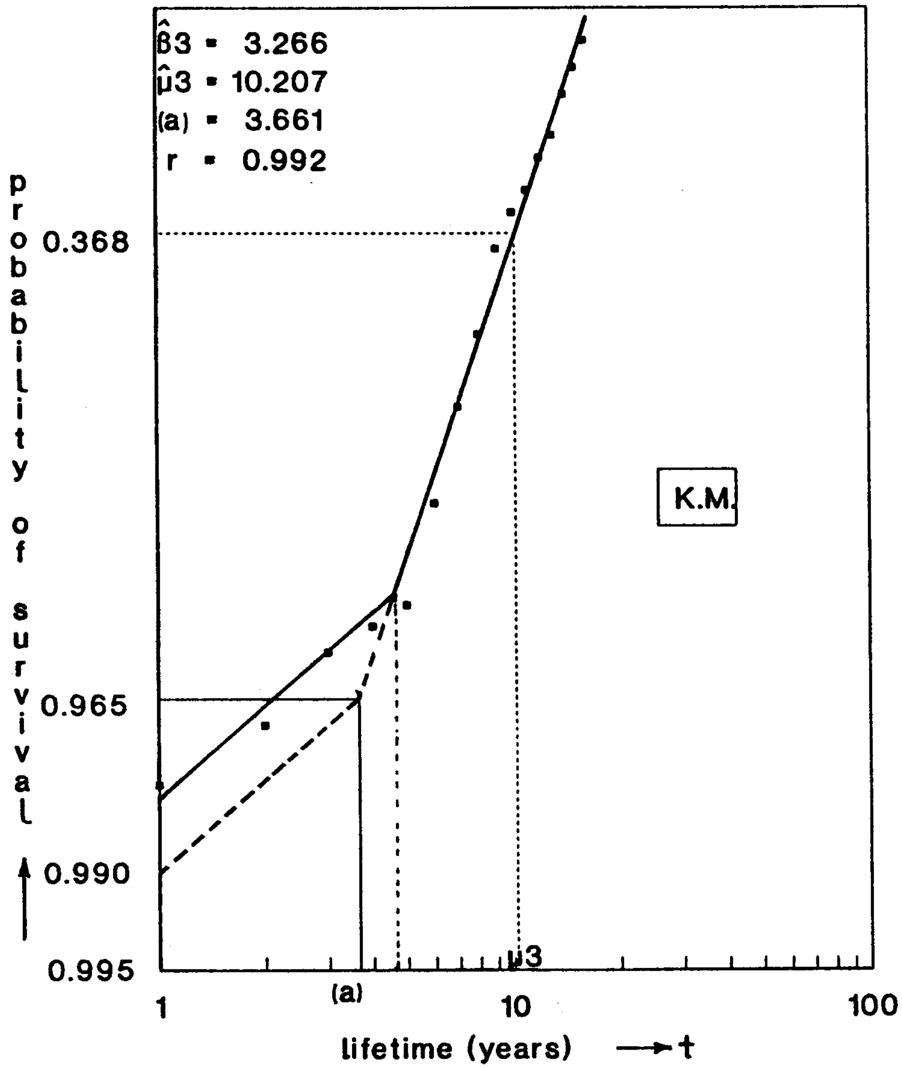


Plot of H-residuals: Machining equipment, Code D.10.2.



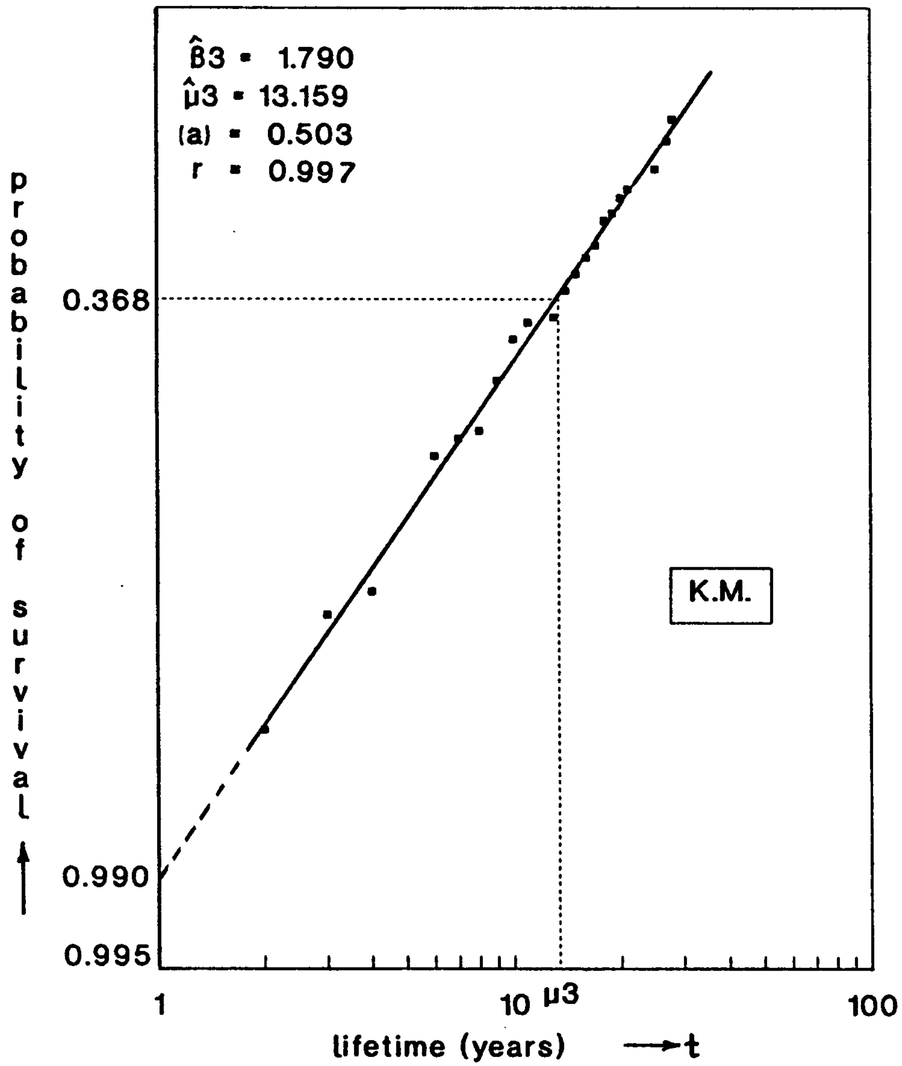
D 1.1 Passenger- & deliverycars

τ_j	d_j	$\hat{S}_j$
1	1	0.986
2	3	0.945
3	5	0.877
4	5	0.808
5	9	0.685
6	16	0.466
7	7	0.370
8	9	0.247
9	11	0.096
10	4	0.041
11	1	0.027
12	1	0.014
19	1	0.000



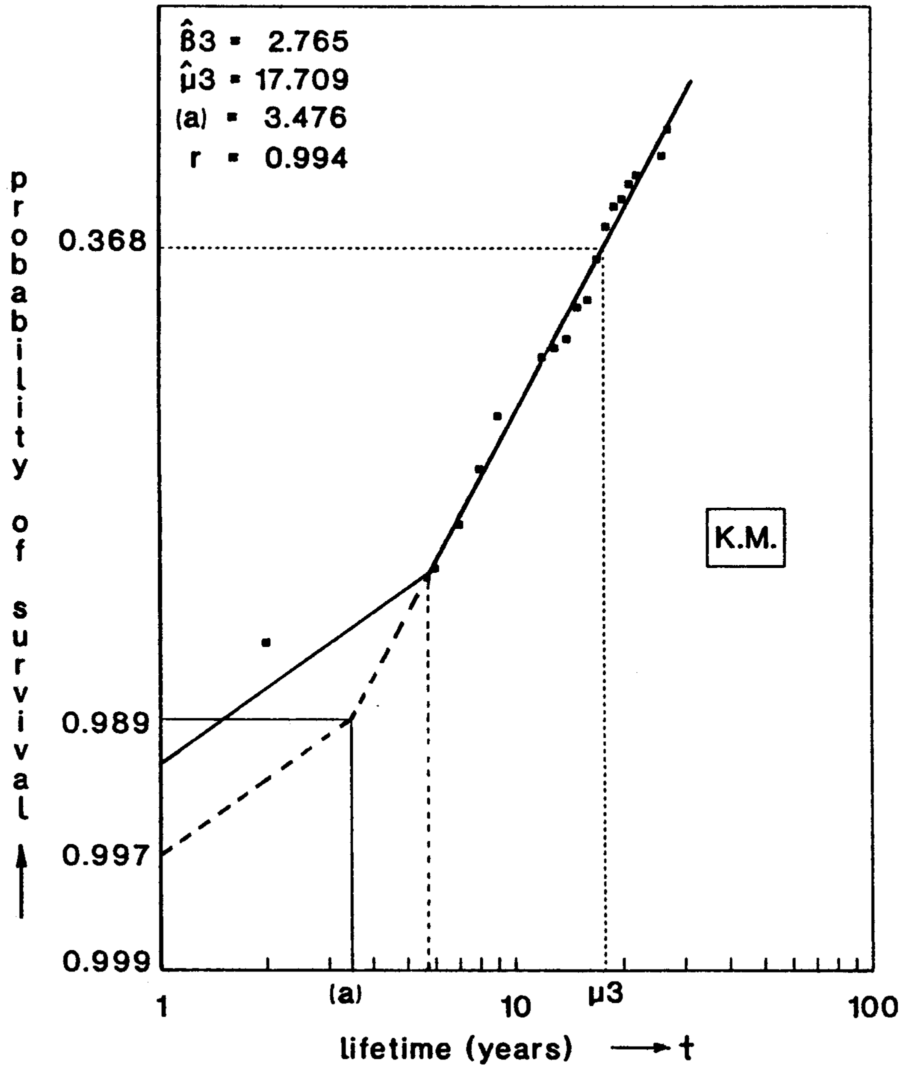
D 2.1 Lorries and trucks

τ_j	d_j	$\hat{S}_j$
1	2	0.981
2	1	0.971
3	2	0.952
4	1	0.943
5	1	0.933
6	7	0.867
7	12	0.752
8	14	0.619
9	22	0.410
10	10	0.314
11	6	0.257
12	8	0.181
13	5	0.133
14	7	0.067
15	3	0.038
16	2	0.019
20	2	0.000



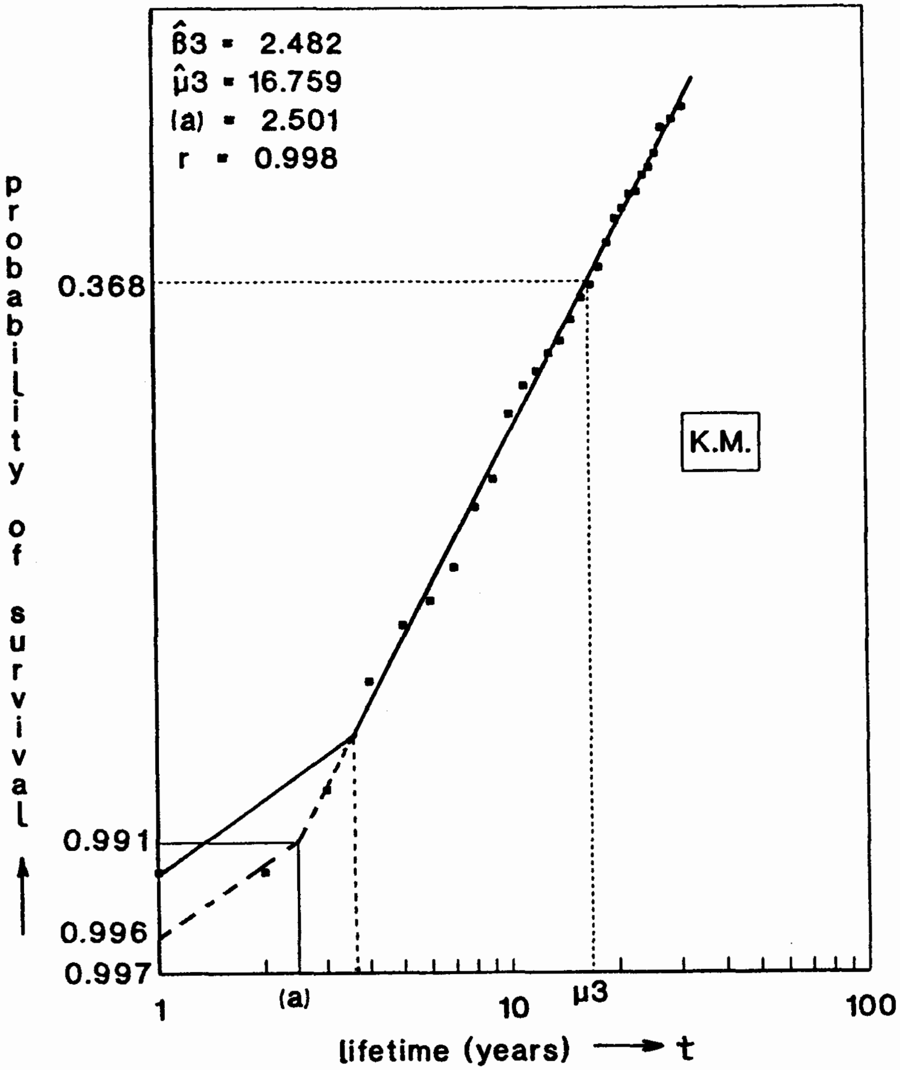
D 5.2 Wrapping equipment

τ_j	d_j	$\hat{S}_j$
2	2	0.969
3	3	0.922
4	1	0.906
6	10	0.750
7	2	0.719
8	1	0.703
9	7	0.594
10	7	0.484
11	3	0.438
13	1	0.422
14	5	0.344
15	3	0.297
16	3	0.250
17	2	0.219
18	4	0.156
19	1	0.141
20	2	0.109
21	1	0.094
25	2	0.063
27	2	0.031
28	1	0.016
35	1	0.000



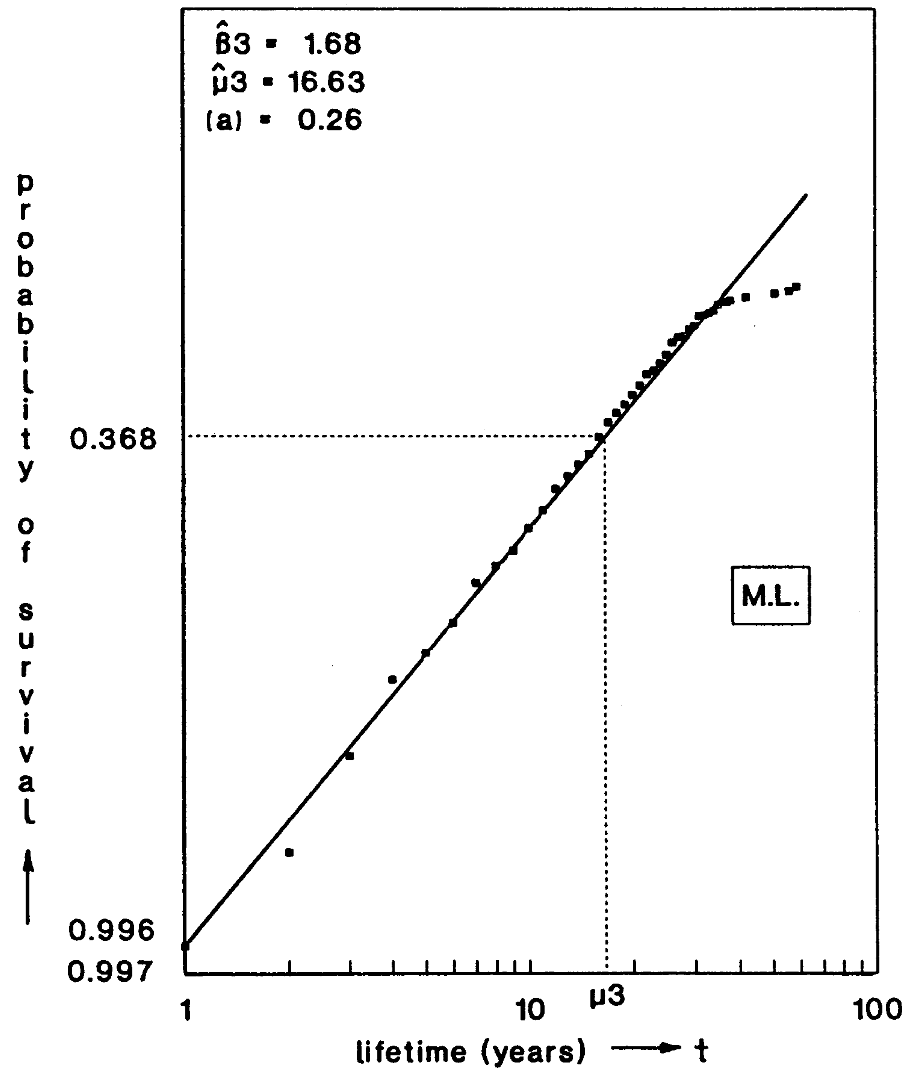
D 6.3 Pumps & compressors

T_j	d_j	$\hat{S}_j$
2	1	0.977
6	1	0.955
7	1	0.932
8	2	0.886
9	3	0.818
12	5	0.705
13	1	0.682
14	1	0.659
15	4	0.568
16	1	0.545
17	6	0.409
18	5	0.295
19	3	0.227
20	1	0.205
21	2	0.159
22	1	0.136
26	2	0.091
27	2	0.045
29	1	0.023
31	1	0.000



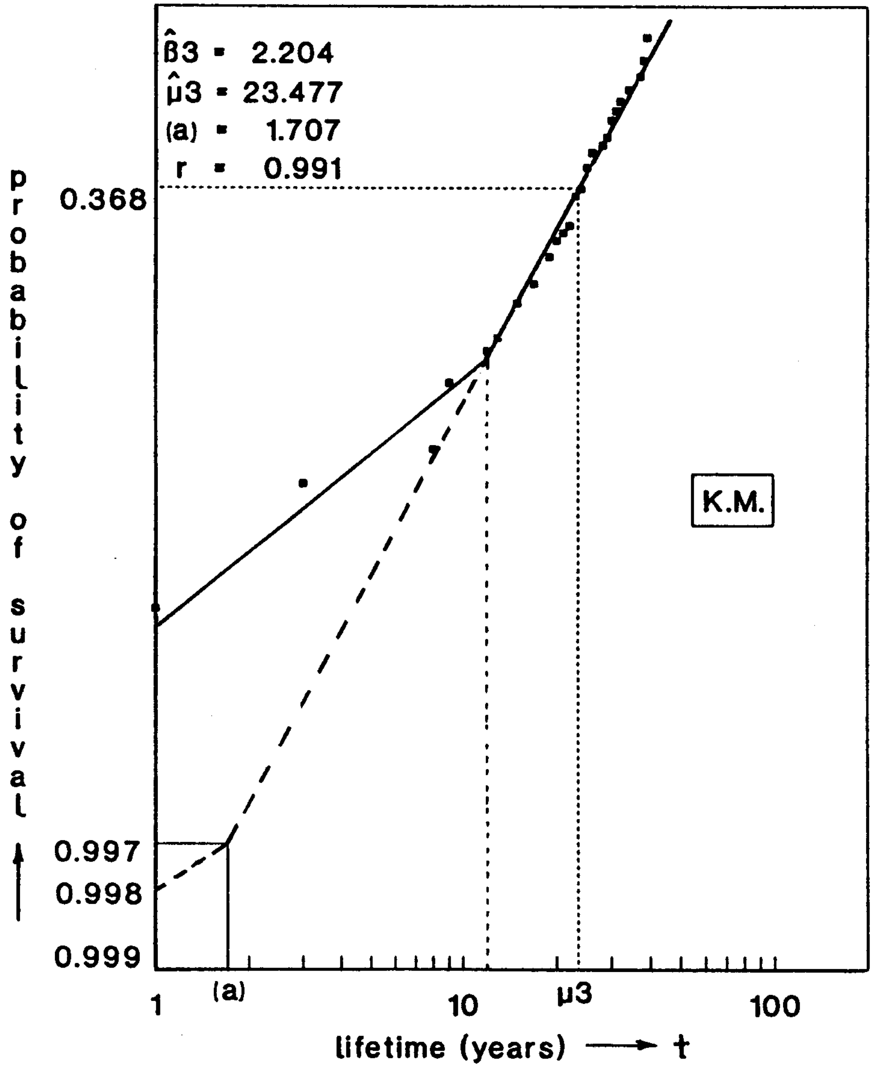
D 7.1 Electric generators

τ_j	d_j	$\hat{S}_j$
1	1	0.993
3	1	0.987
4	3	0.976
5	3	0.947
6	2	0.933
7	3	0.913
8	8	0.860
9	5	0.827
10	16	0.720
11	9	0.660
12	5	0.627
13	7	0.580
14	5	0.547
15	9	0.487
16	10	0.420
17	6	0.380
18	8	0.327
19	11	0.253
20	10	0.187
21	4	0.160
22	5	0.127
23	1	0.120
24	5	0.087
25	2	0.073
26	3	0.053
27	4	0.027
29	1	0.020
31	1	0.013
32	2	0.000



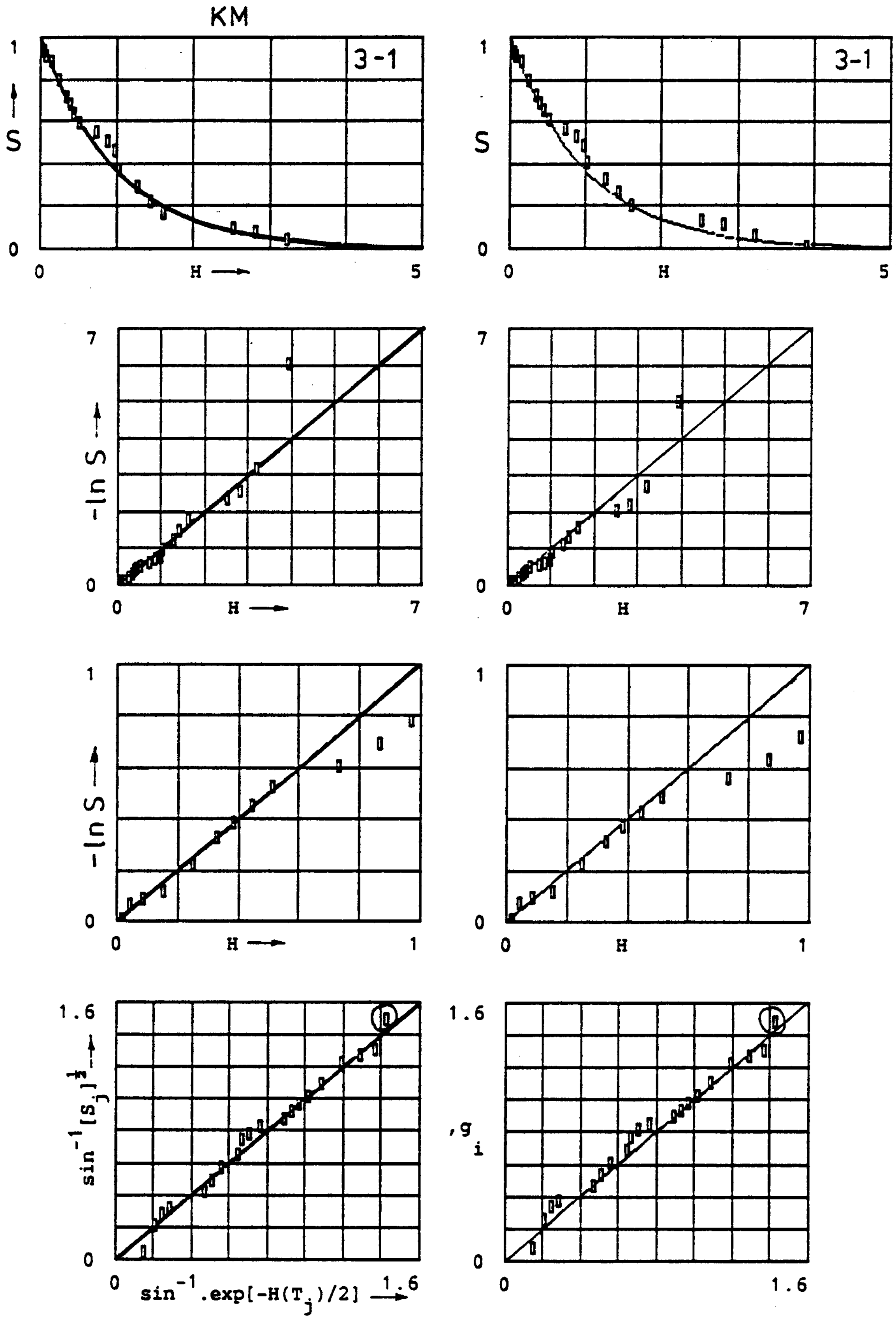
D 9.1 Measuring & controlling equipment

T_j	d_j	$\hat{S}_j$	T_j	d_j	$\hat{S}_j$
1	2	0.997	22	21	0.147
2	4	0.990	23	6	0.136
3	12	0.969	24	12	0.116
4	22	0.932	25	13	0.094
5	13	0.910	26	16	0.066
6	19	0.877	27	6	0.056
7	35	0.818	28	1	0.055
8	19	0.785	29	6	0.044
9	20	0.751	30	3	0.039
10	33	0.695	31	7	0.027
11	31	0.642	32	1	0.026
12	41	0.572	33	1	0.024
13	27	0.526	34	1	0.022
14	25	0.484	35	3	0.017
15	25	0.441	37	1	0.015
16	39	0.375	38	1	0.014
17	35	0.315	42	1	0.012
18	21	0.279	51	1	0.010
19	18	0.249	56	1	0.009
20	20	0.215	59	1	0.007
21	19	0.182	83	4	0.000

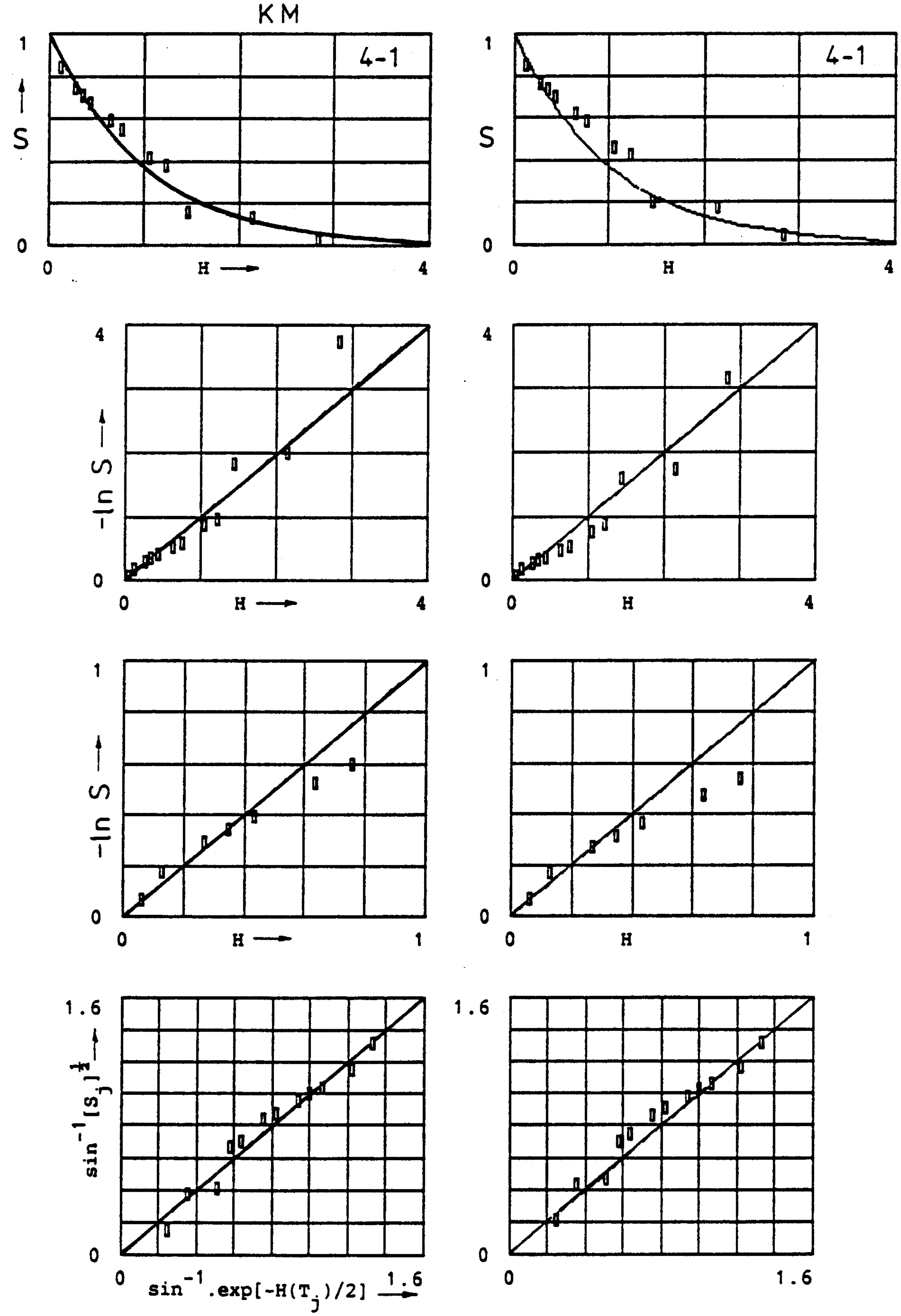


D 10.2 machining equipment

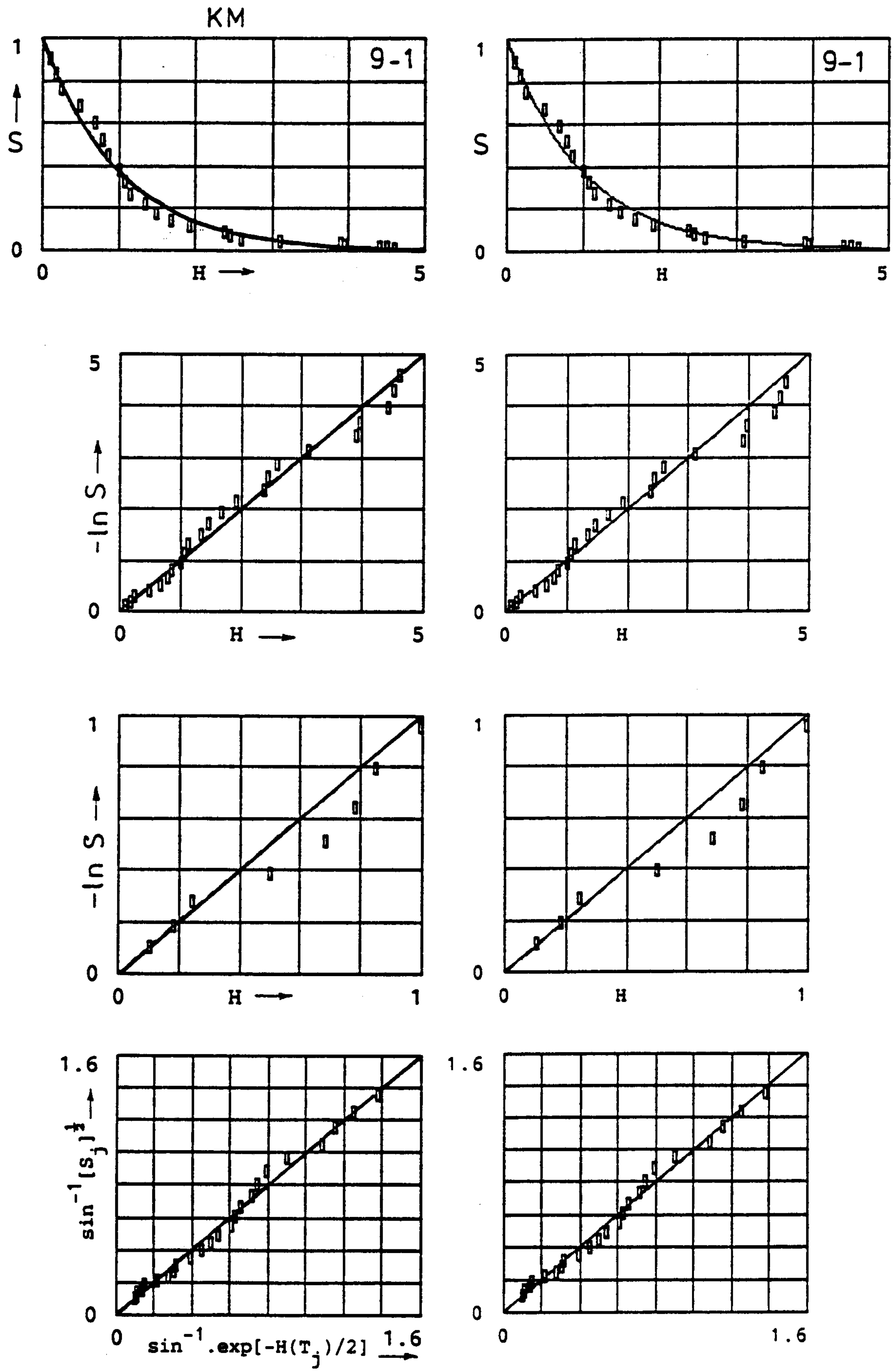
τ_j	d_j	$\hat{S}_j$
1	1	0.977
3	2	0.930
8	1	0.907
9	3	0.837
12	2	0.791
14	1	0.767
15	3	0.698
17	2	0.651
19	3	0.581
20	2	0.535
21	1	0.512
22	1	0.488
23	4	0.395
24	1	0.372
25	3	0.302
26	2	0.256
28	1	0.233
29	1	0.209
30	2	0.163
31	1	0.140
32	1	0.116
34	1	0.093
37	1	0.070
38	1	0.046
39	1	0.023
40	1	0.000



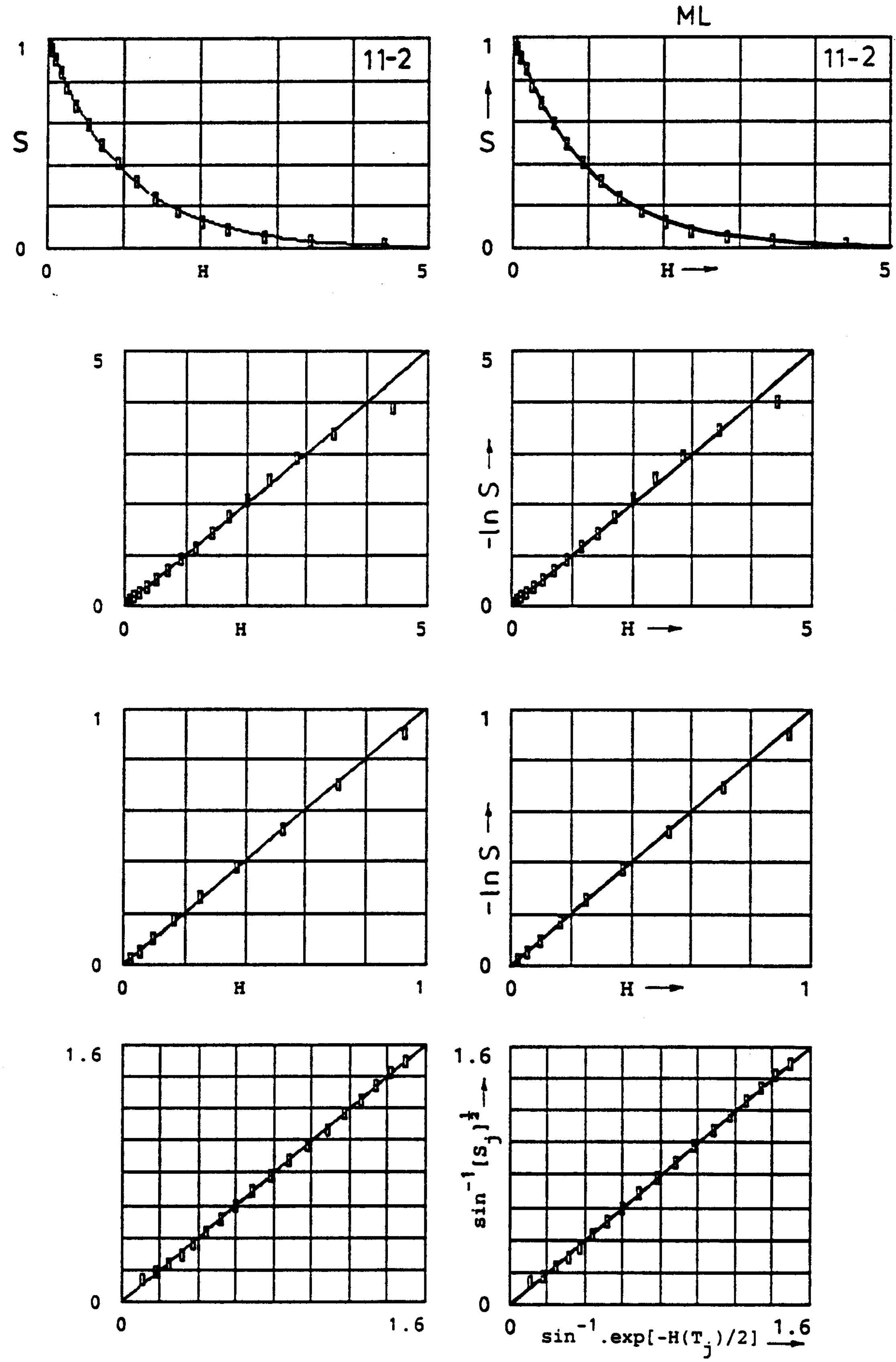
Plot of H-residuals: Water work pumps, Code 3-1



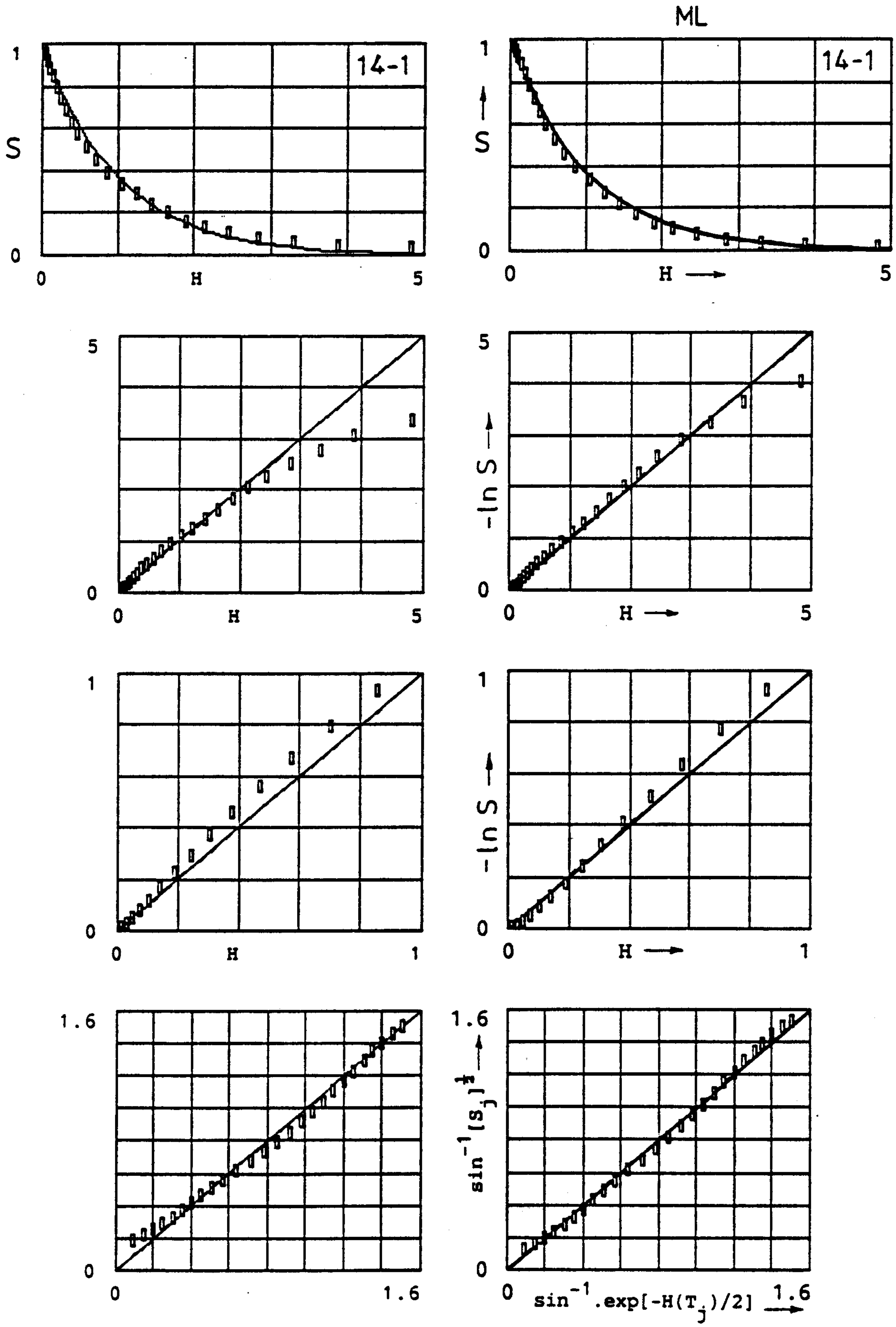
Plot of H-residuals: Water works steam engines, Code 4-1



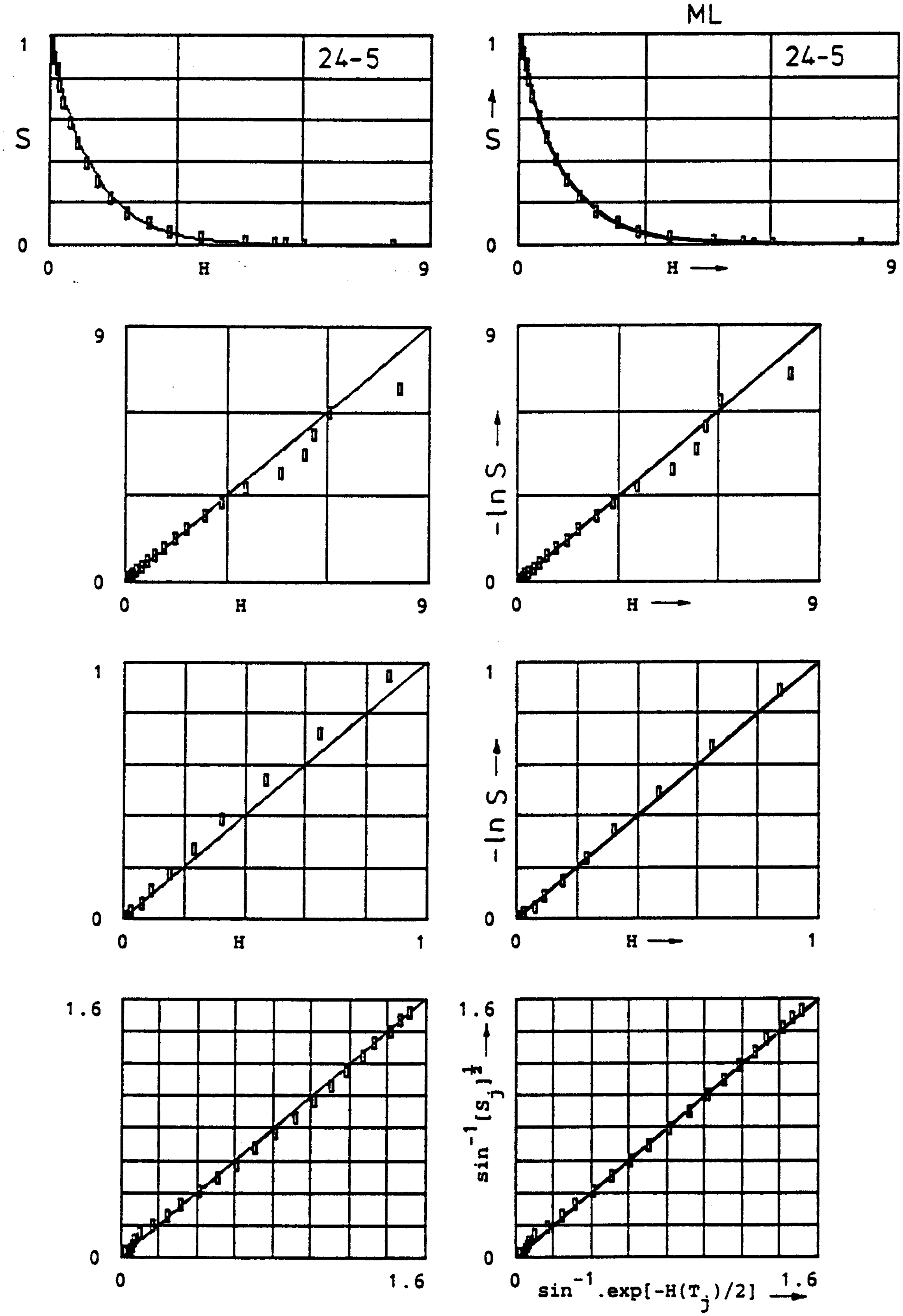
Plot of H-residuals: Central office equipment (telephone), Code 9-1



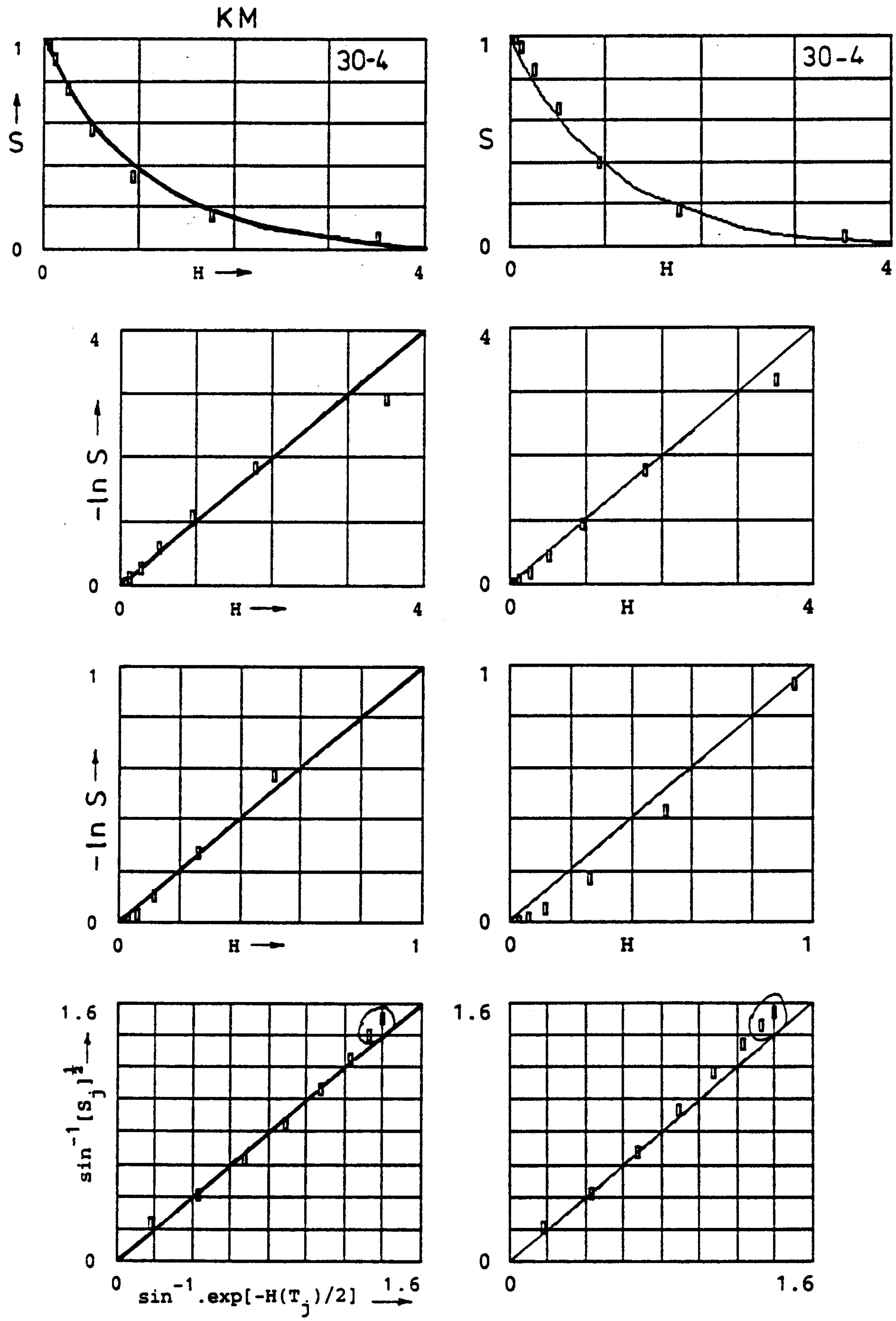
Plot of H-residuals: Aerial cables (telephone), Code 11-2



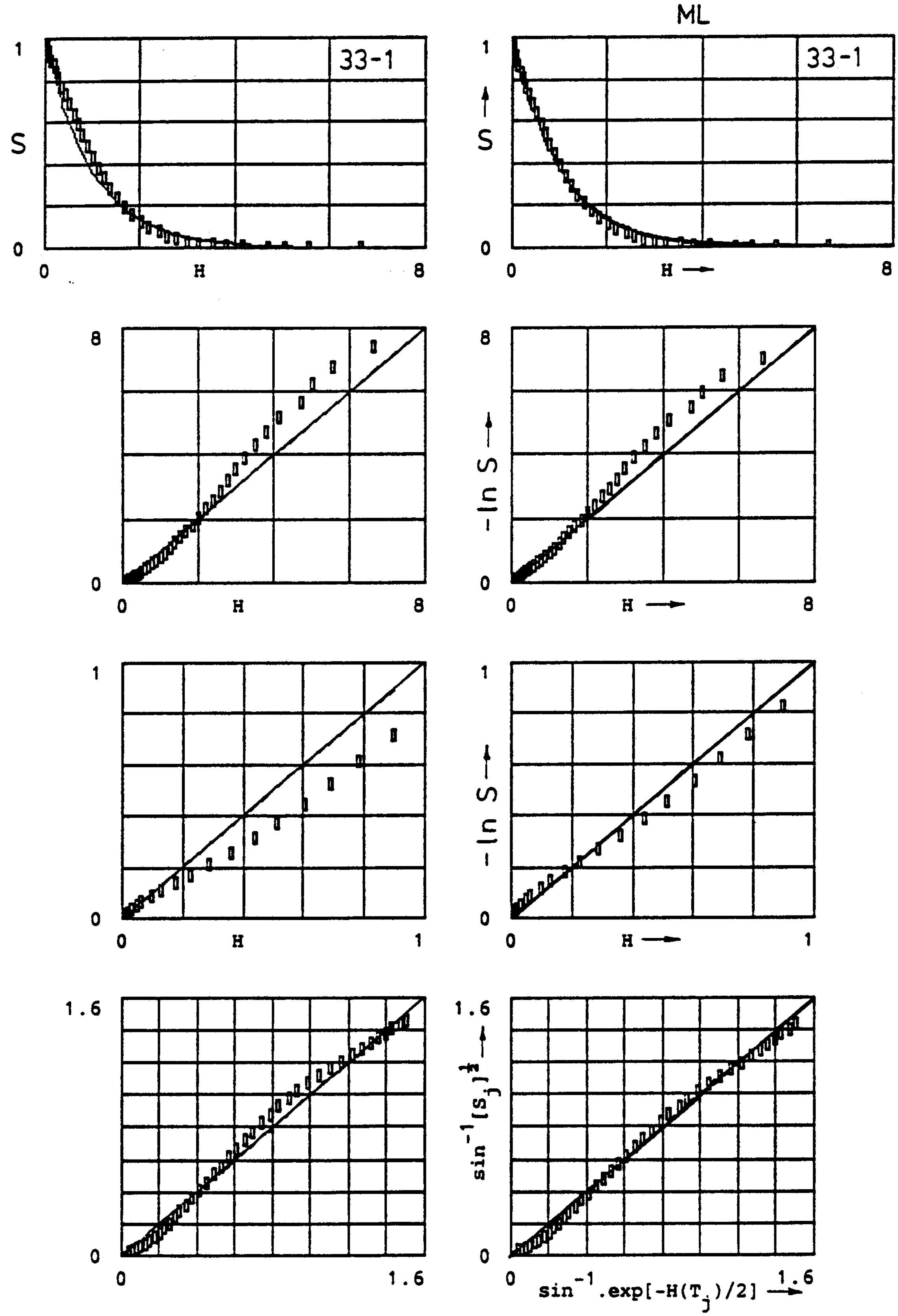
Plot of H-residuals: Underground cables (telephone), Code 14-1



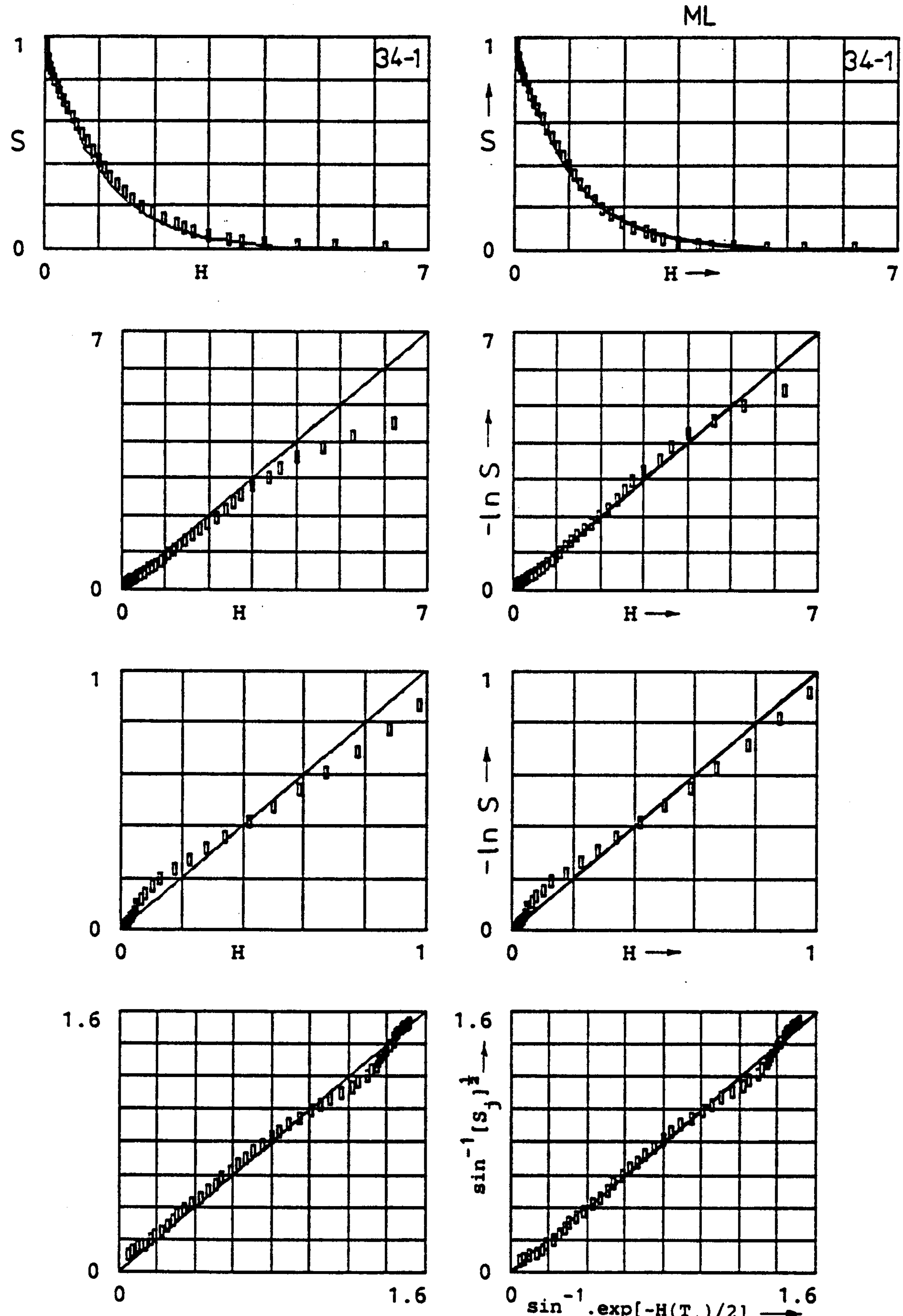
Plot of H-residuals: Wooden poles (telegraph), Code 24-5



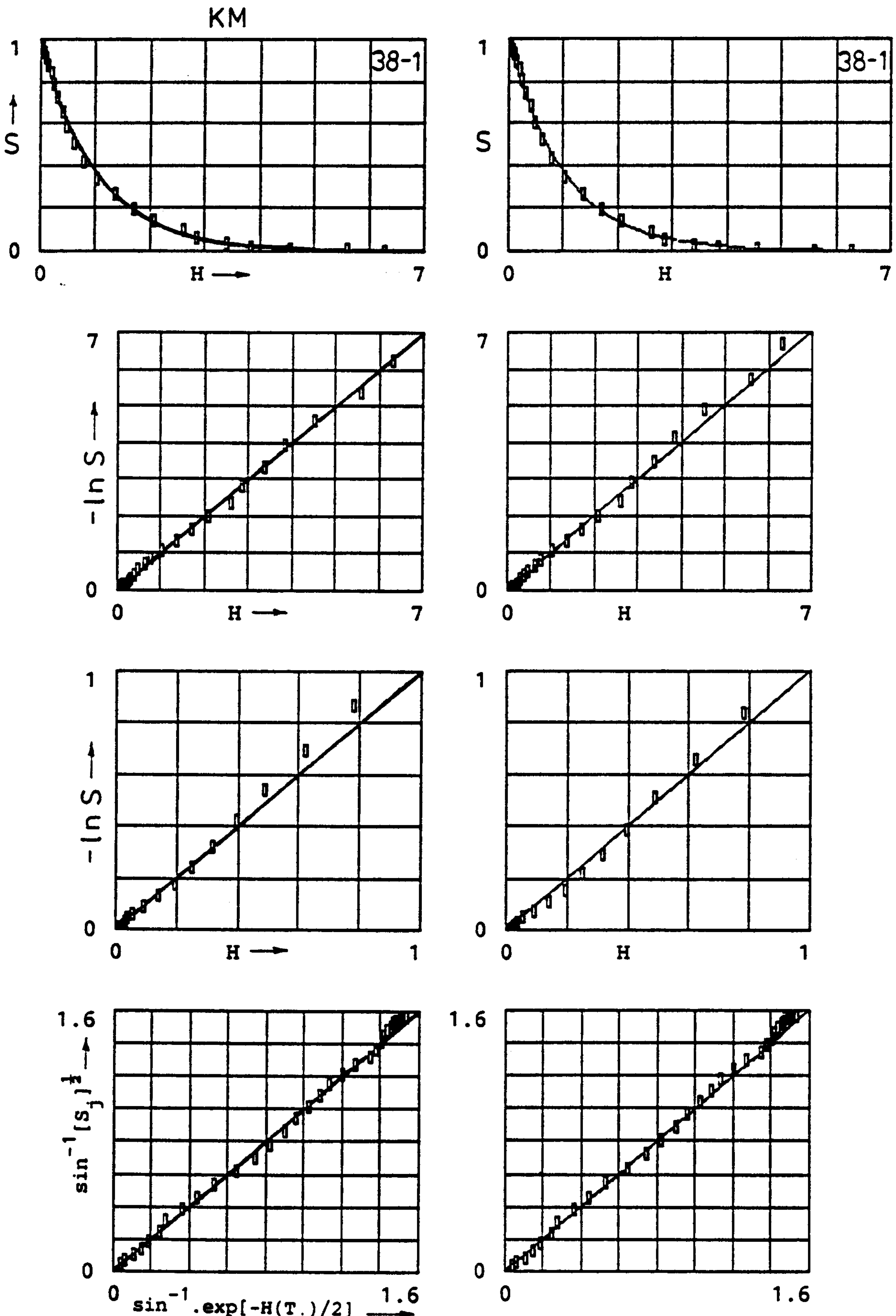
Plot of H-residuals: Mazda B-lamps (60W electric), Code 30-4



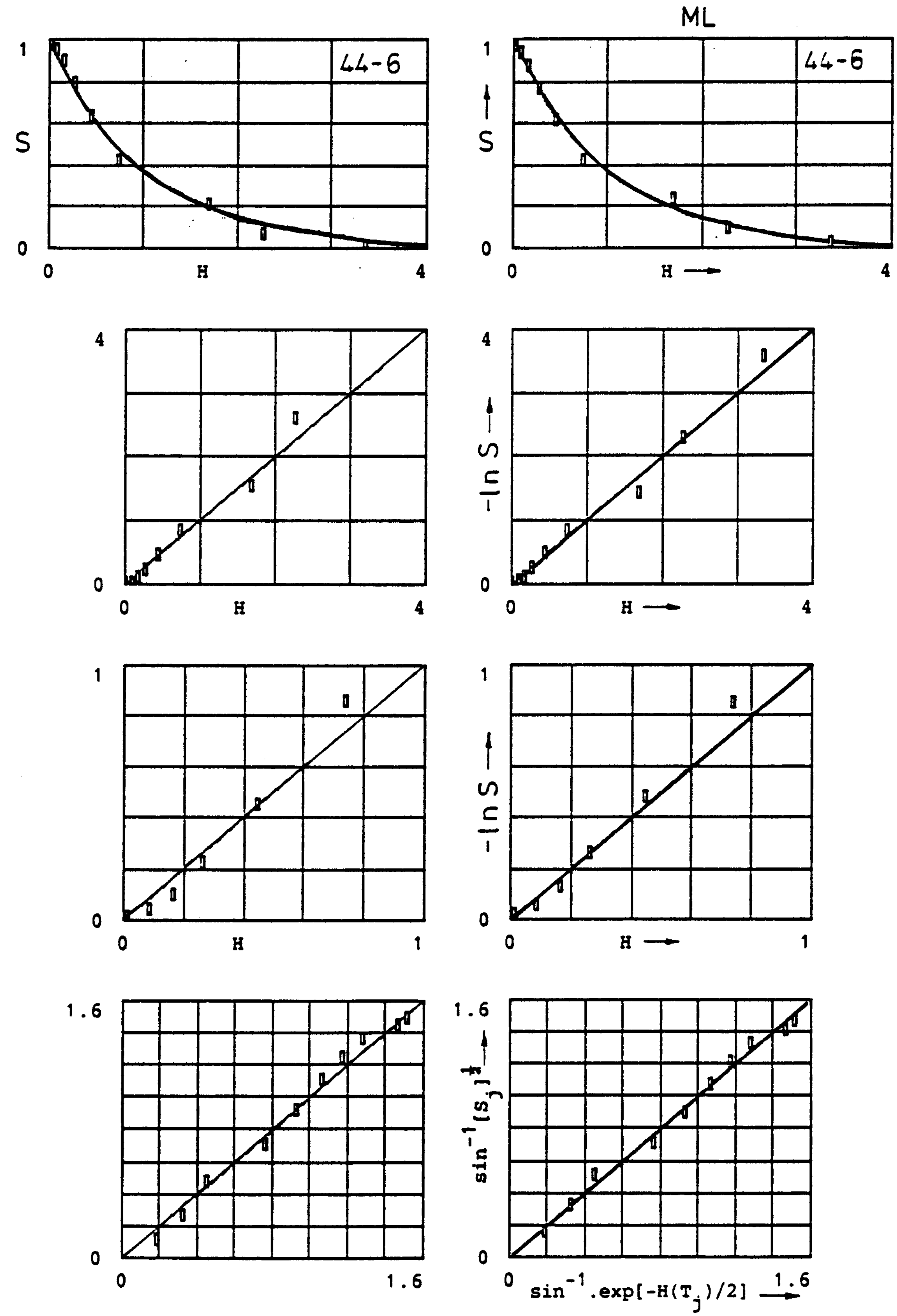
Plot of H-residuals: Steam locomotives (rail road), Code 33-1



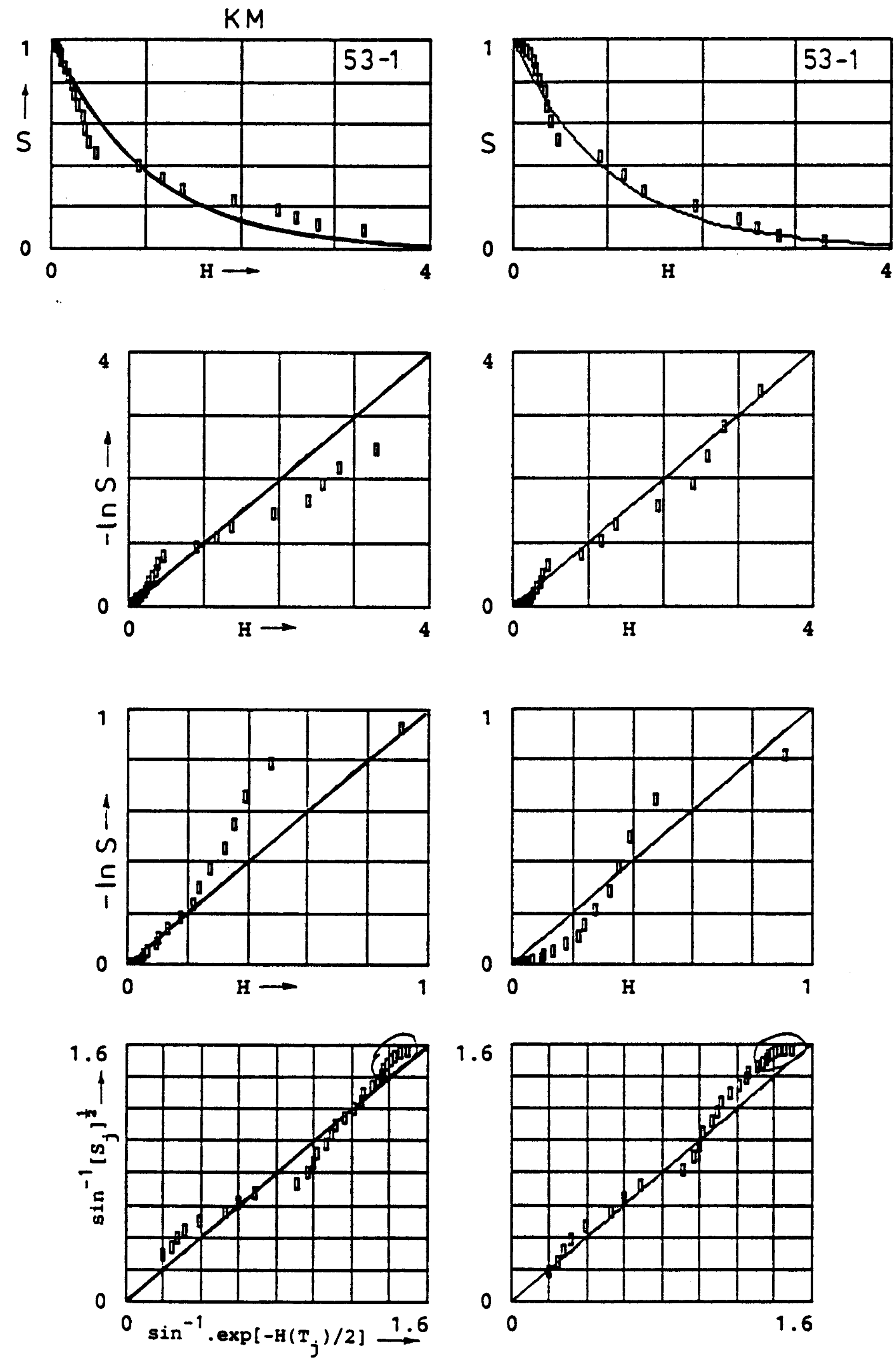
Plot of H-residuals: Passenger cars (rail road), Code 34-1



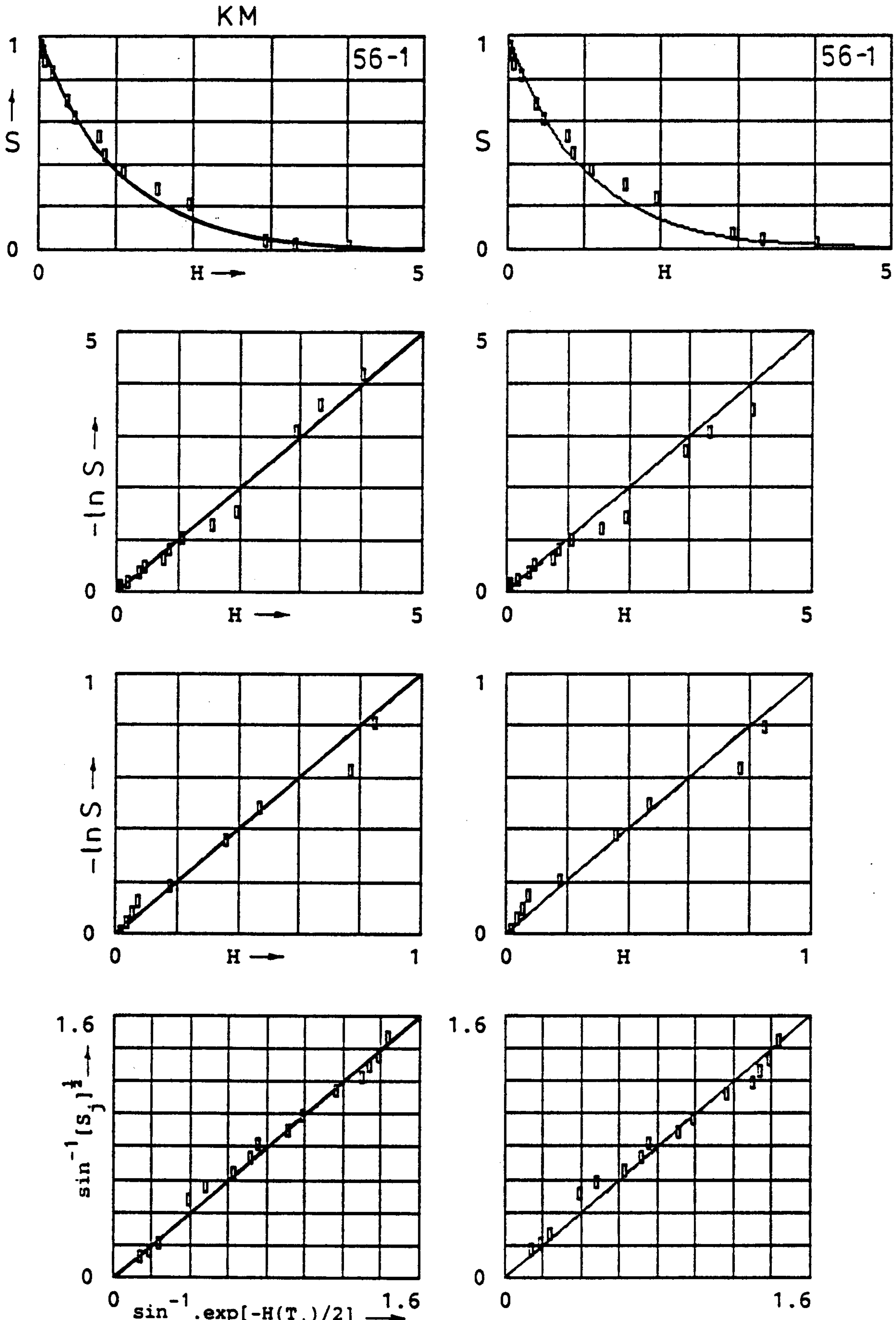
Plot of H-residuals: Coal flat cars (rail road), Code 38-1



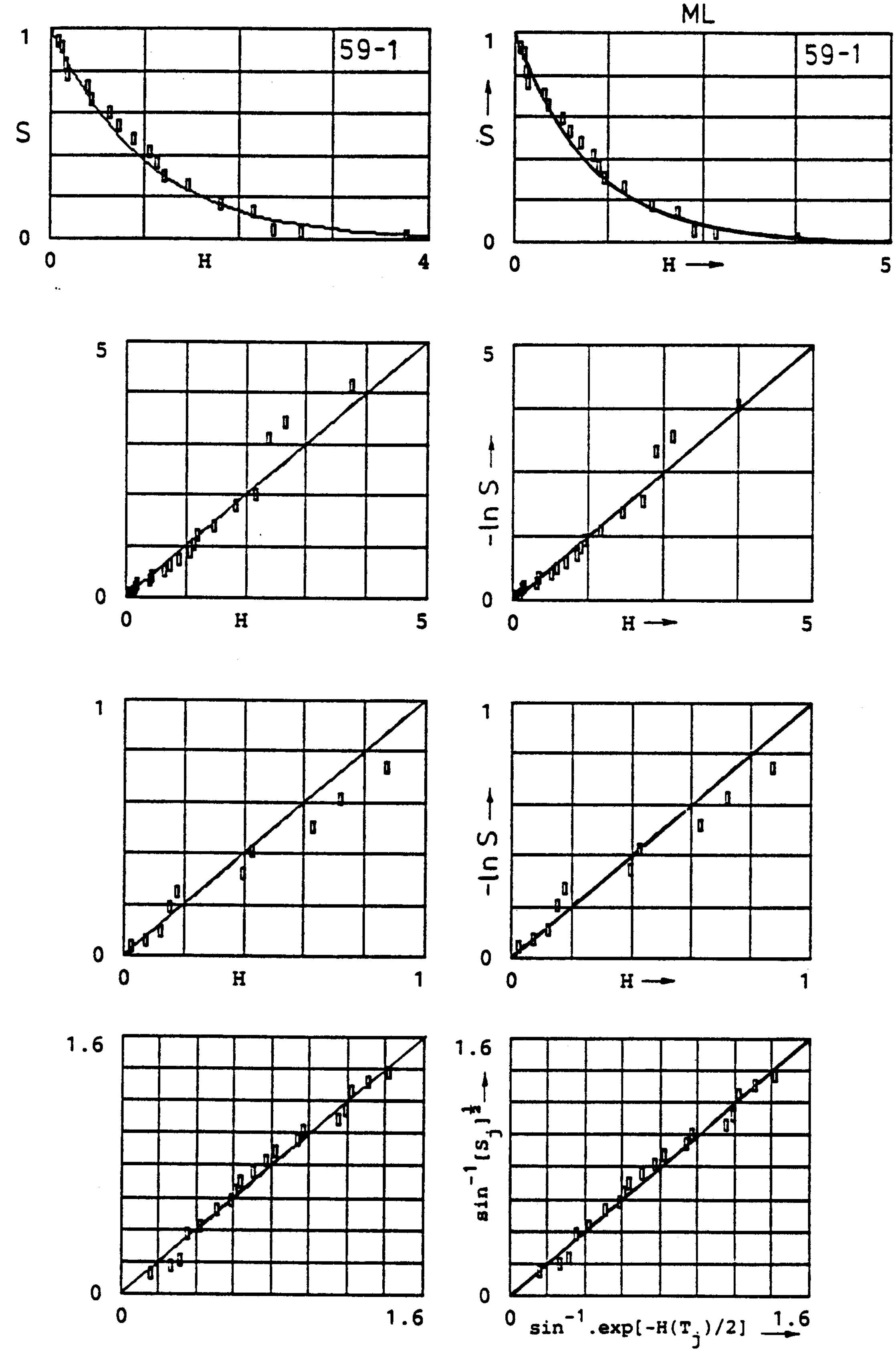
Plot of H-residuals: Crossties (rail road), Code 44-6



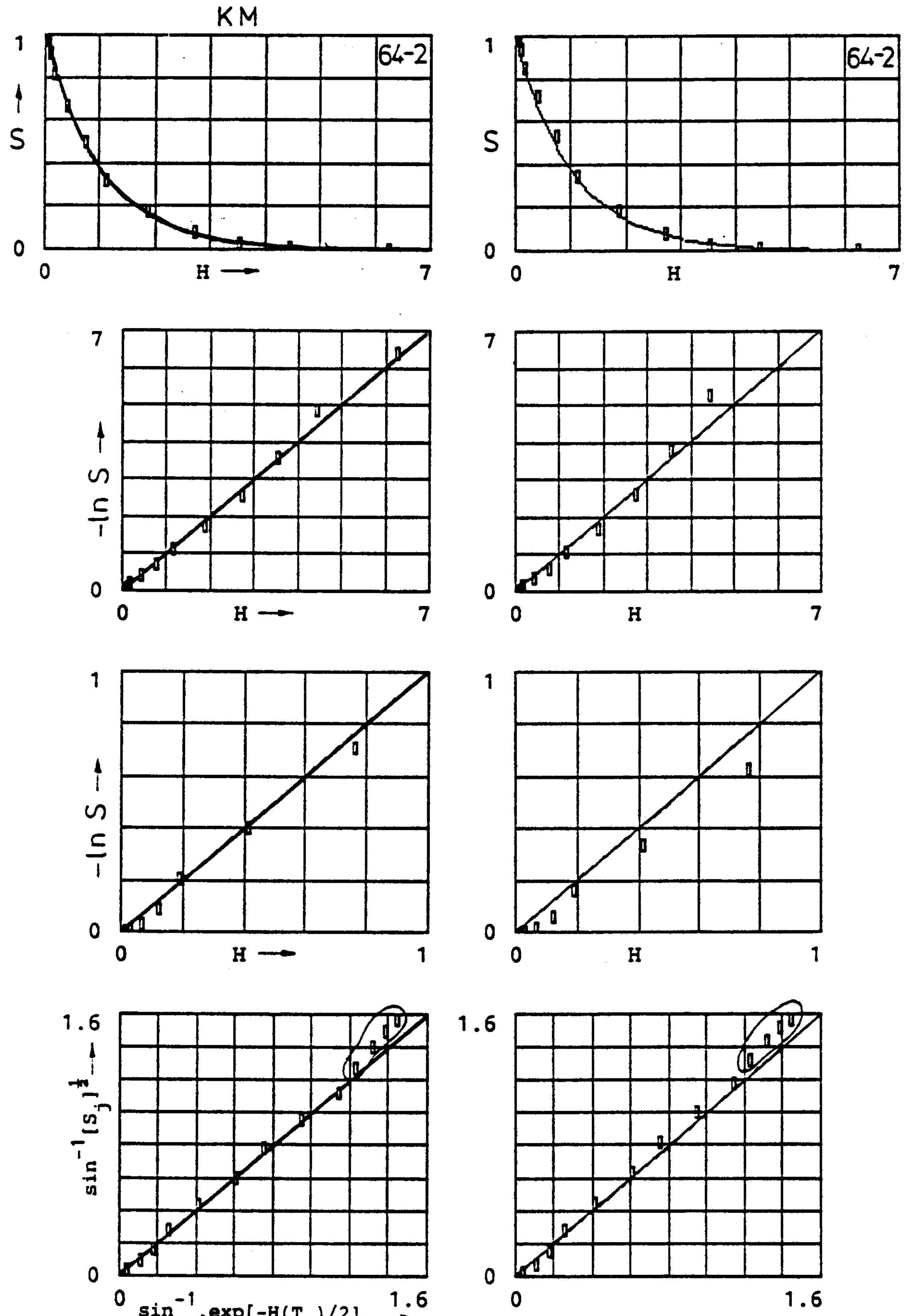
Plot of H-residuals: Rodger ballast cars (rail road), Code 53-1



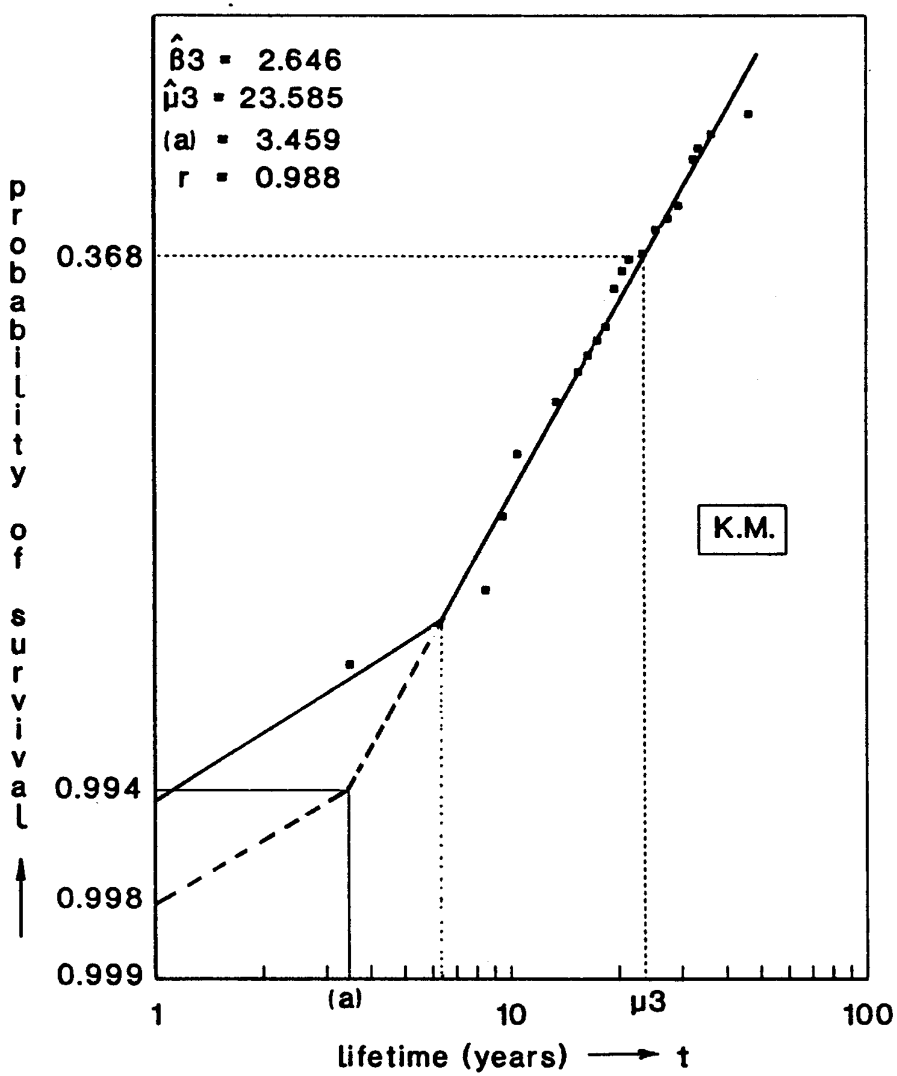
Plot of H-residuals: Corn cultivators (1-row), Code 56-1



Plot of H-residuals: Grain binders (5 to 8-foot), Code 59-1

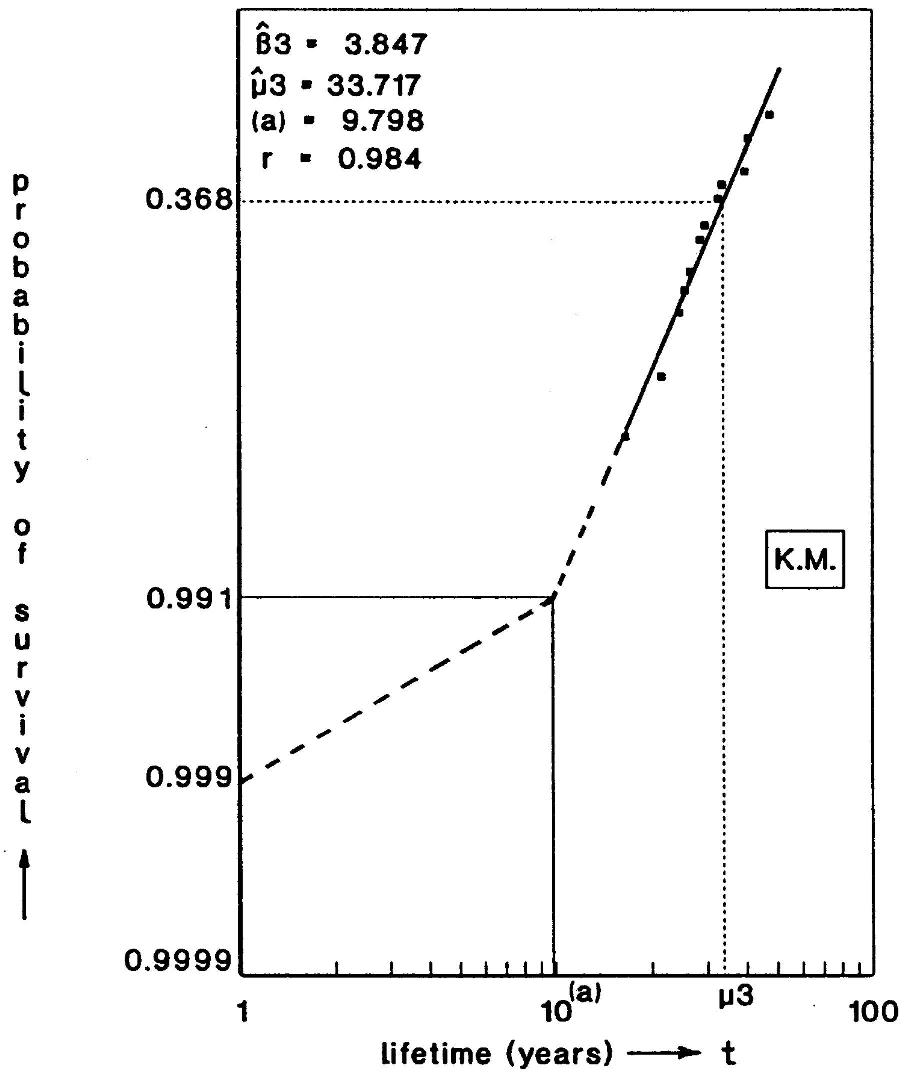


Plot of H-residuals: Passenger automobiles (1922), Code 64-2



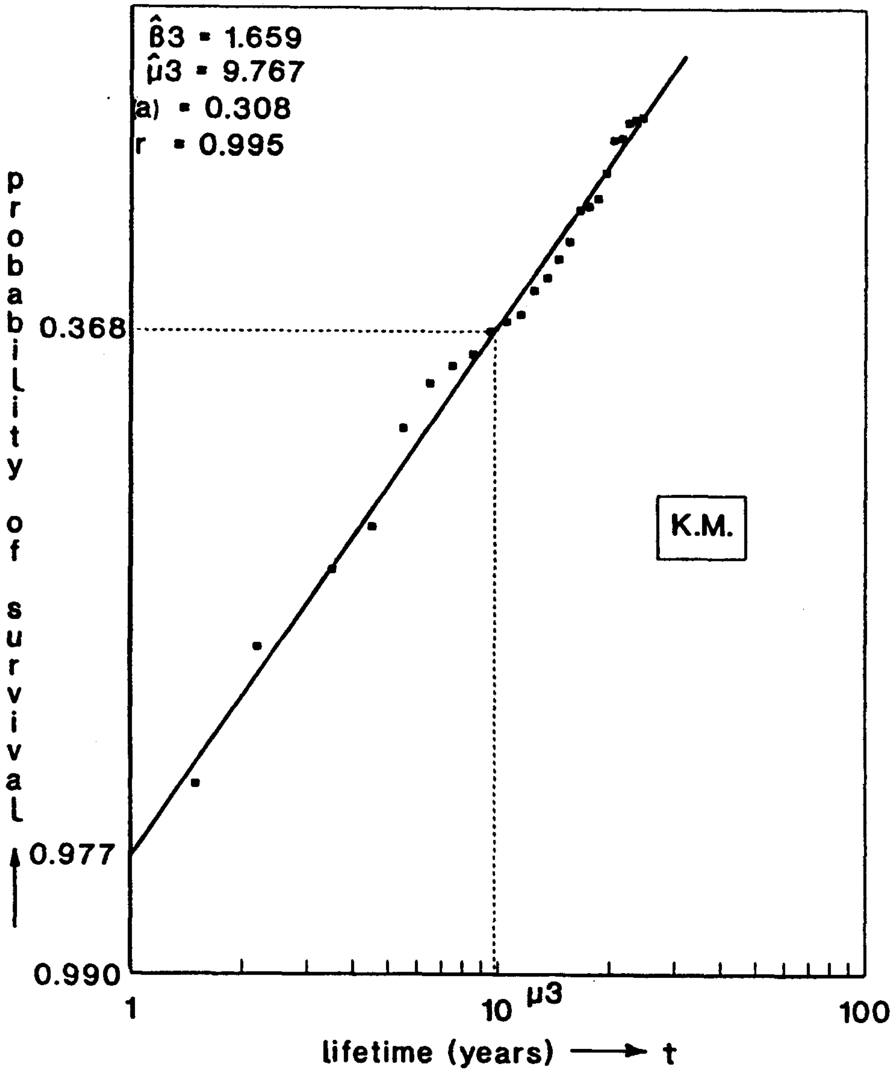
3-1 water work pumps

T_j	d_j	$\hat{S}_j$
3.5	1	0.980
8.5	1	0.960
9.5	2	0.920
10.5	3	0.860
13.5	4	0.780
15.5	3	0.720
16.5	2	0.680
17.5	2	0.640
18.5	2	0.600
19.5	6	0.480
20.5	3	0.420
21.5	2	0.380
23.5	1	0.360
25.5	4	0.280
27.5	2	0.240
29.5	2	0.200
32.5	6	0.080
33.5	1	0.060
36.5	1	0.040
46.5	1	0.020
50.5	1	0.000



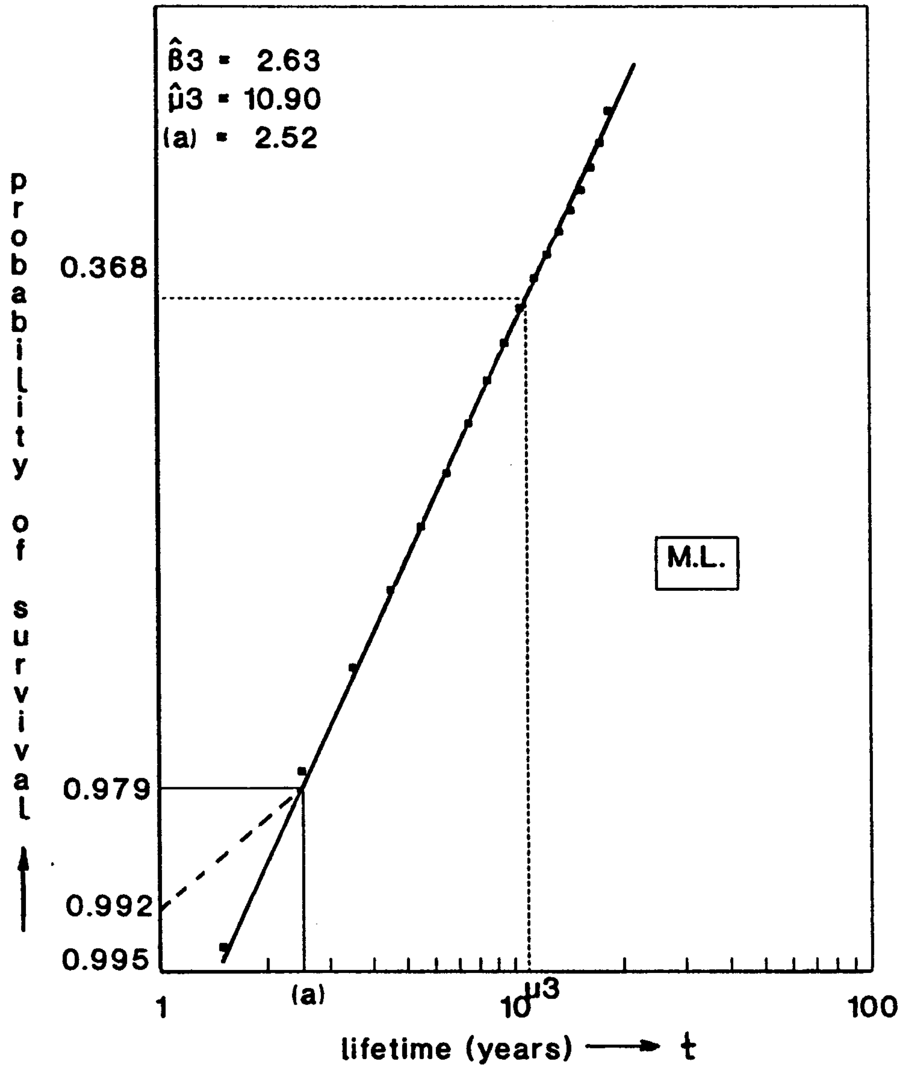
4-1 steam engines

T_j	d_j	$\hat{S}_j$
16.5	1	0.941
21.5	1	0.882
24.5	2	0.765
25.5	1	0.706
26.5	1	0.647
28.5	2	0.529
29.5	1	0.471
32.5	2	0.353
33.5	1	0.294
39.5	1	0.235
40.5	2	0.118
47.5	1	0.059
51.5	1	0.000



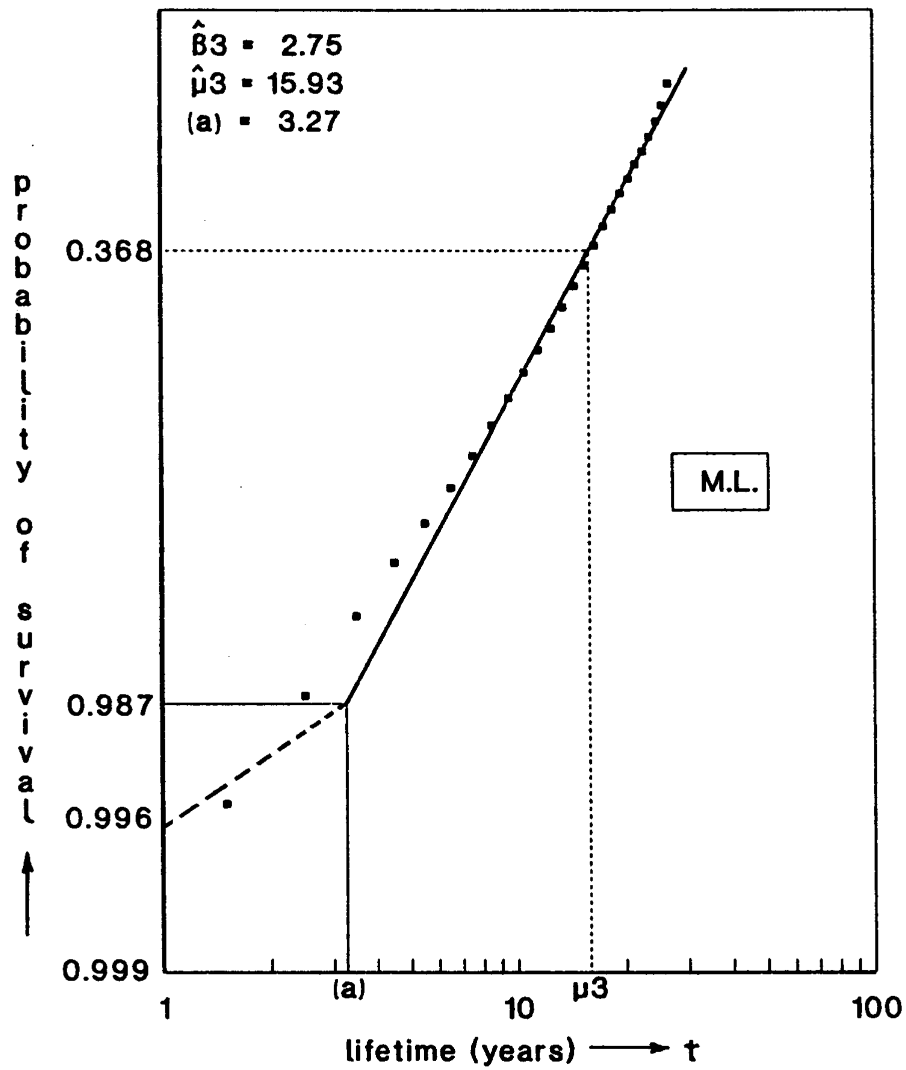
9-1 Central office equipment

τ_j	d_j	$\hat{S}_j$
1.5	38	0.962
2.5	60	0.902
3.5	67	0.835
4.5	53	0.782
5.5	174	0.608
6.5	104	0.504
7.5	45	0.459
8.5	30	0.429
9.5	60	0.369
10.5	27	0.342
11.5	19	0.323
12.5	62	0.261
13.5	31	0.230
14.5	44	0.186
15.5	38	0.148
16.5	55	0.093
17.5	6	0.087
18.5	12	0.075
19.5	30	0.045
20.5	25	0.020
21.5	1	0.019
22.5	7	0.012
23.5	1	0.011
24.5	1	0.010
25.5	10	0.000



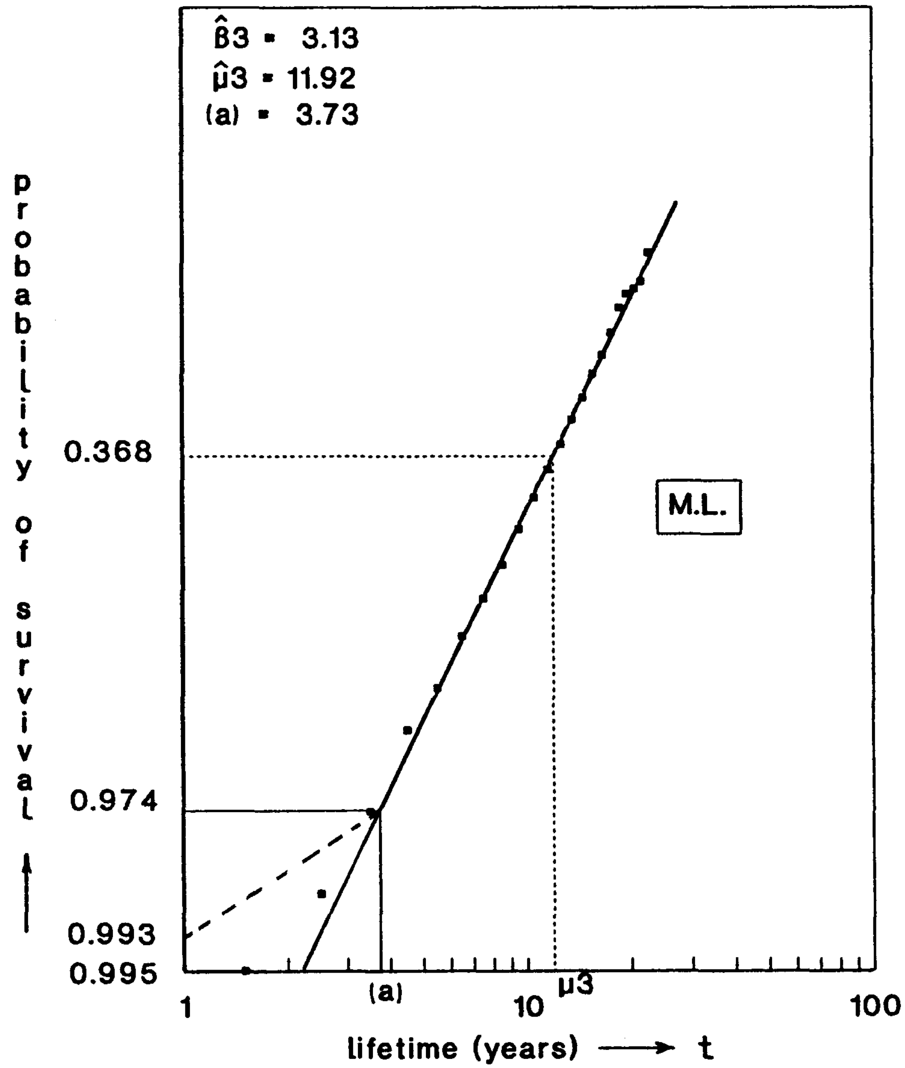
11-2 Aerial Cables

T_j	d_j	$\hat{S}_j$
1.5	6	0.994
2.5	18	0.976
3.5	29	0.947
4.5	42	0.905
5.5	57	0.848
6.5	71	0.777
7.5	88	0.689
8.5	96	0.593
9.5	99	0.494
10.5	99	0.395
11.5	85	0.310
12.5	68	0.242
13.5	59	0.183
14.5	49	0.134
15.5	40	0.094
16.5	35	0.059
17.5	27	0.032
18.5	20	0.012
19.5	12	0.000



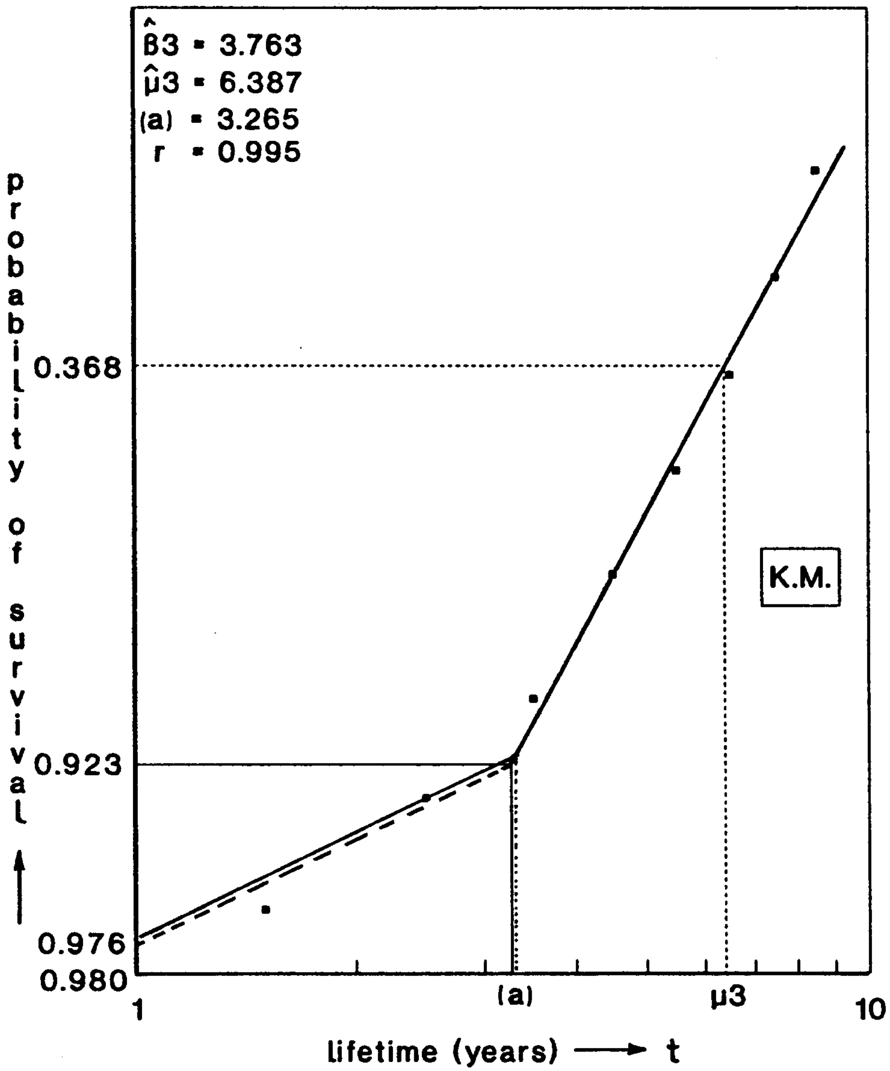
14-1 Underground Cable

τ_j	d_j	$\hat{S}_j$
1.5	5	0.995
2.5	9	0.986
3.5	16	0.970
4.5	19	0.951
5.5	21	0.930
6.5	28	0.902
7.5	33	0.869
8.5	40	0.829
9.5	45	0.785
10.5	50	0.735
11.5	52	0.684
12.5	58	0.626
13.5	64	0.563
14.5	69	0.495
15.5	71	0.425
16.5	70	0.356
17.5	65	0.291
18.5	55	0.239
19.5	51	0.190
20.5	41	0.151
21.5	35	0.118
22.5	27	0.087
23.5	25	0.058
24.5	19	0.037
25.5	14	0.021
26.5	11	0.008
27.5	7	0.000



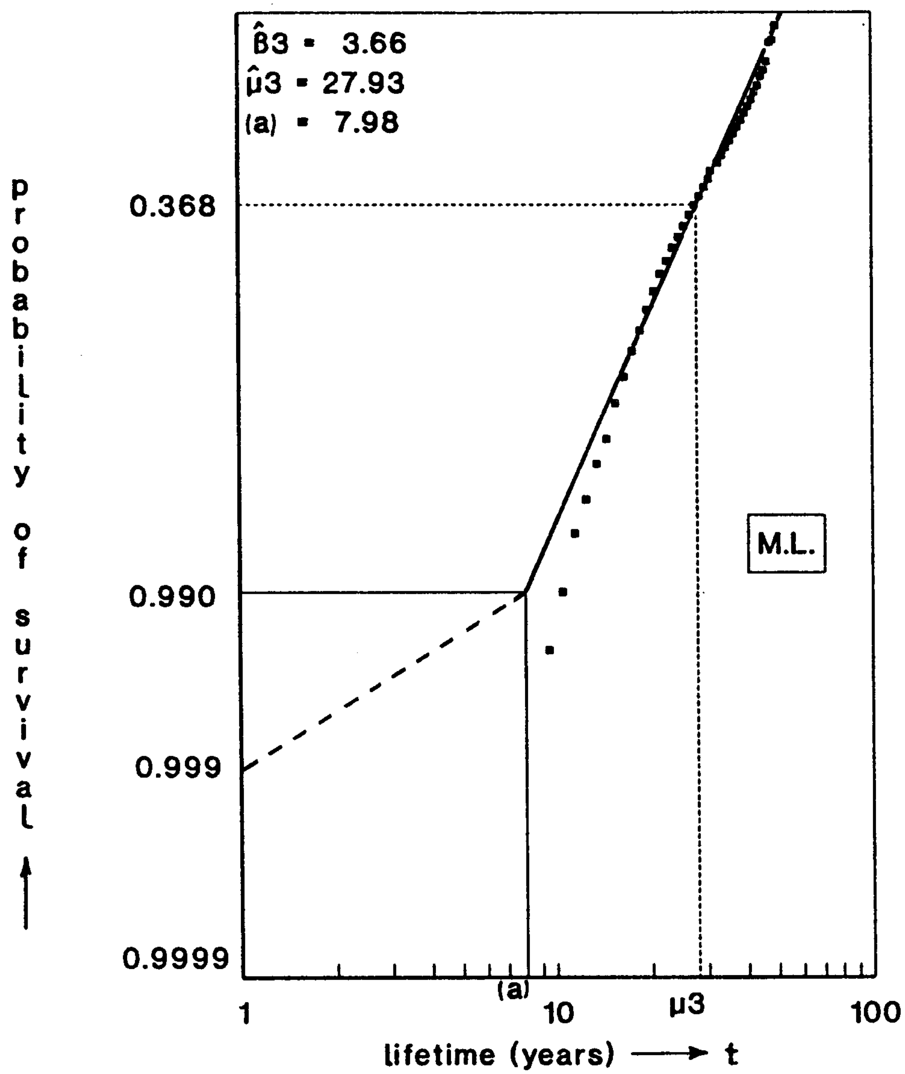
24-5 Wooden poles

T_j	d_j	$\hat{S}_j$
1.5	105	0.9970
2.5	222	0.9890
3.5	435	0.9750
4.5	957	0.9430
5.5	888	0.9130
6.5	1705	0.8560
7.5	1834	0.7950
8.5	2185	0.7220
9.5	2890	0.6260
10.5	3112	0.5220
11.5	3100	0.4190
12.5	2863	0.3240
13.5	2719	0.2330
14.5	2179	0.1600
15.5	1933	0.0960
16.5	1131	0.0580
17.5	903	0.0280
18.5	552	0.0100
19.5	150	0.0050
20.5	36	0.0040
21.5	39	0.0020
22.5	63	0.0003
23.5	9	0.0000



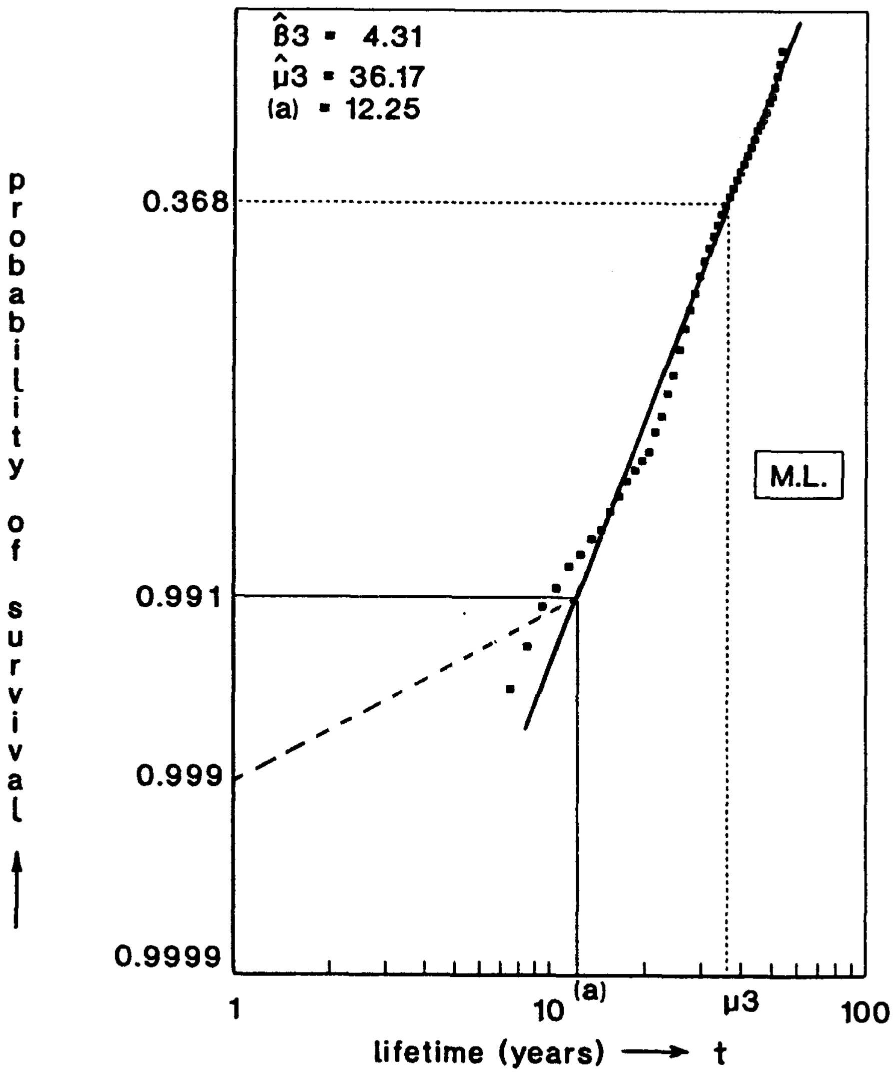
30-4 Mazda B lamps (60 Watt)

τ_j	d_j	$\hat{S}_j$
1.5	3	0.970
2.5	3	0.940
3.5	5	0.890
4.5	12	0.770
5.5	17	0.600
6.5	21	0.390
7.5	22	0.170
8.5	14	0.030
9.5	3	0.000



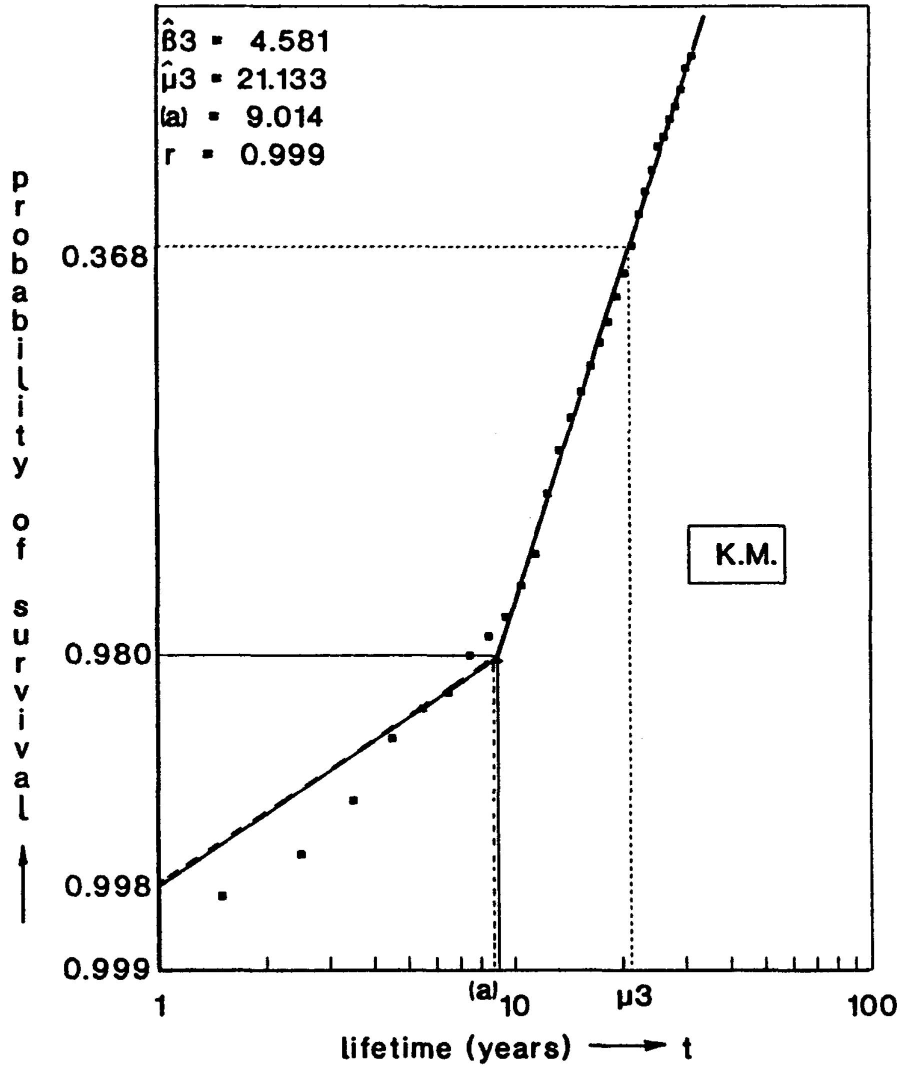
33-1 steam locomotives

T_j	d_j	$\hat{S}_j$	T_j	d_j	$\hat{S}_j$
9.5	4	0.995	29.5	30	0.287
10.5	4	0.990	30.5	27	0.252
11.5	8	0.979	31.5	25	0.220
12.5	8	0.969	32.5	26	0.187
13.5	12	0.954	33.5	21	0.160
14.5	12	0.938	34.5	20	0.134
15.5	23	0.909	35.5	16	0.113
16.5	23	0.879	36.5	17	0.091
17.5	31	0.839	37.5	12	0.076
18.5	31	0.799	38.5	10	0.063
19.5	39	0.749	39.5	10	0.050
20.5	39	0.699	40.5	8	0.040
21.5	43	0.644	41.5	8	0.030
22.5	34	0.600	42.5	6	0.022
23.5	41	0.547	43.5	5	0.015
24.5	35	0.502	44.5	5	0.009
25.5	35	0.457	45.5	2	0.006
26.5	38	0.408	46.5	2	0.004
27.5	33	0.366	47.5	2	0.001
28.5	31	0.326	50.5	1	0.000



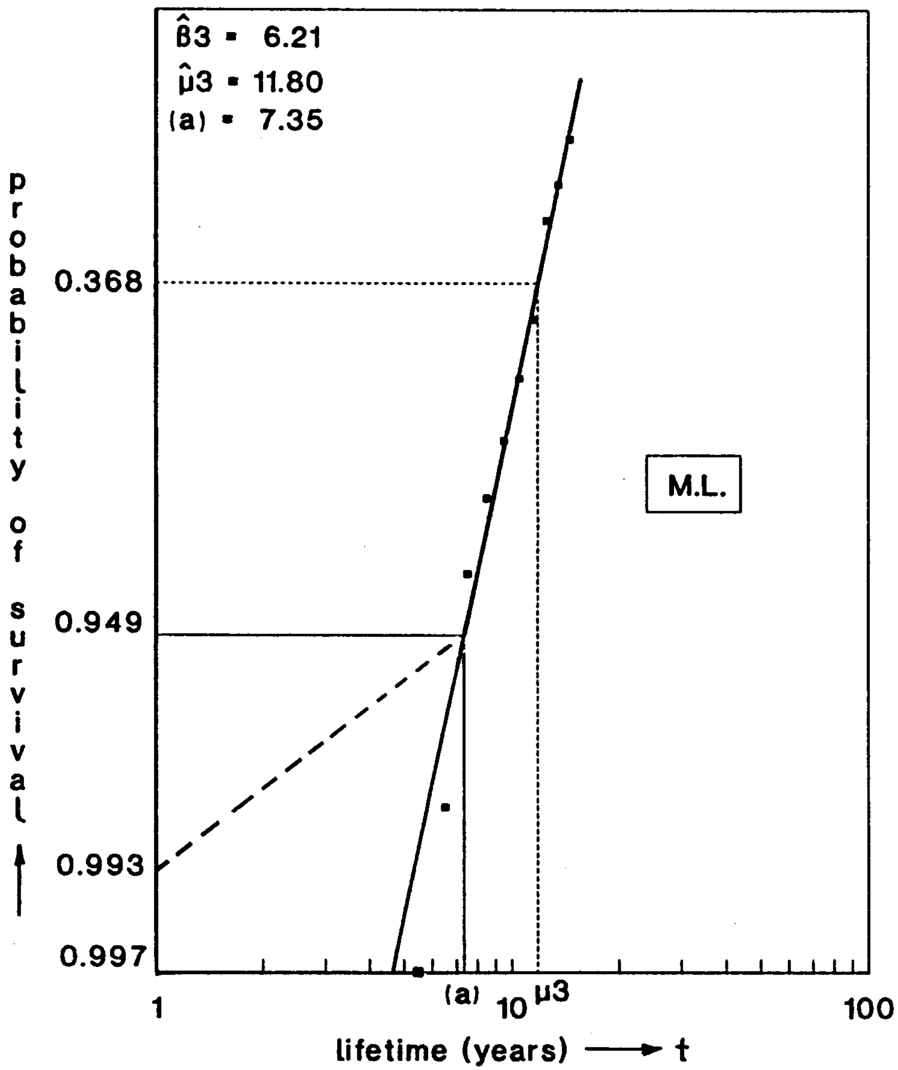
34-1 Passenger cars

r_j	d_j	$\hat{s}_j$	r_j	d_j	$\hat{s}_j$
7.5	45	0.997	31.5	726	0.557
8.5	30	0.995	32.5	711	0.510
9.5	45	0.992	33.5	726	0.462
10.5	30	0.990	34.5	711	0.415
11.5	45	0.987	35.5	575	0.377
12.5	30	0.985	36.5	560	0.340
13.5	45	0.982	37.5	575	0.302
14.5	30	0.980	38.5	560	0.265
15.5	76	0.975	39.5	499	0.232
16.5	76	0.970	40.5	484	0.200
17.5	76	0.965	41.5	454	0.170
18.5	76	0.960	42.5	454	0.140
19.5	76	0.955	43.5	378	0.115
20.5	76	0.950	44.5	378	0.090
21.5	197	0.937	45.5	197	0.077
22.5	182	0.925	46.5	182	0.065
23.5	348	0.902	47.5	227	0.050
24.5	333	0.880	48.5	227	0.035
25.5	605	0.840	49.5	121	0.027
26.5	605	0.800	50.5	136	0.018
27.5	681	0.755	51.5	121	0.010
28.5	681	0.710	52.5	76	0.005
29.5	802	0.657	53.5	45	0.002
30.5	787	0.605	54.5	30	0.000



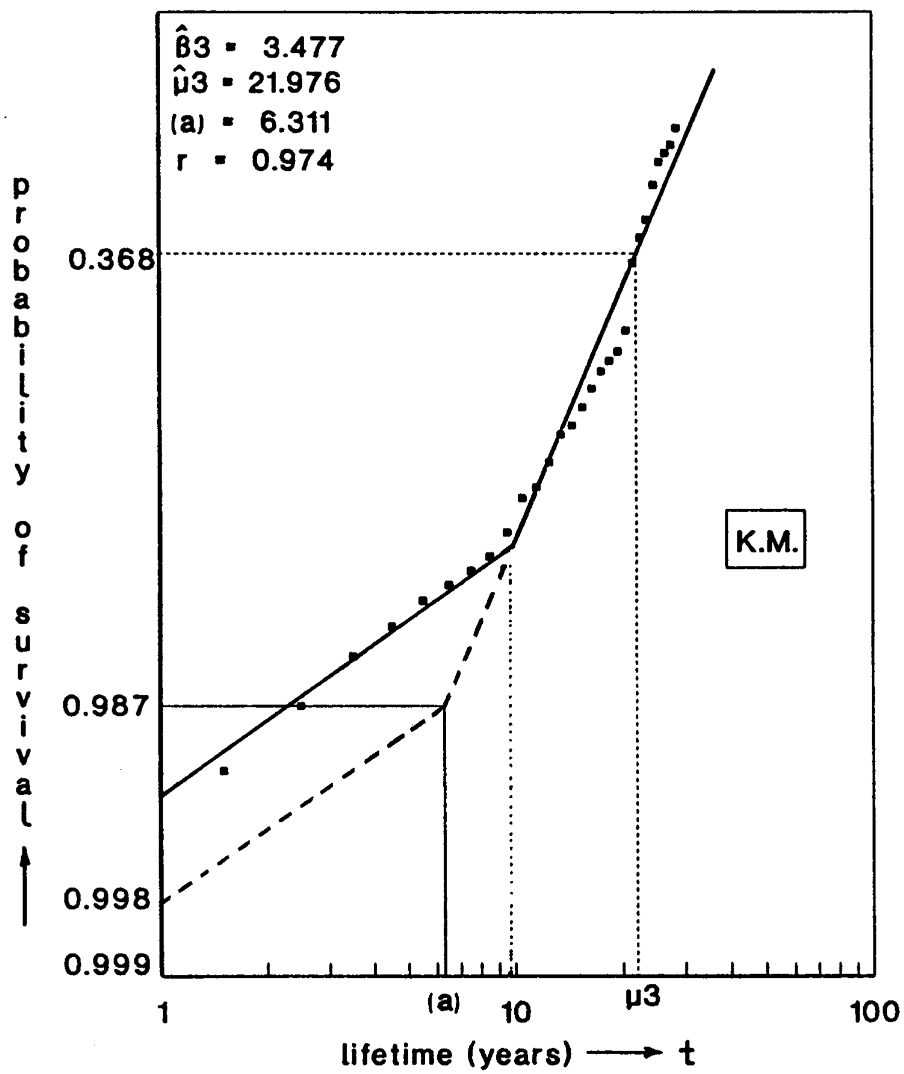
38-1 coal flat cars

τ_j	d_j	$\hat{S}_j$	τ_j	d_j	$\hat{S}_j$
1.5	5	0.998	17.5	149	0.671
2.5	3	0.997	18.5	155	0.614
3.5	5	0.995	19.5	209	0.537
4.5	11	0.991	20.5	212	0.459
5.5	8	0.988	21.5	258	0.364
6.5	5	0.986	22.5	296	0.255
7.5	16	0.980	23.5	195	0.183
8.5	11	0.976	24.5	160	0.124
9.5	14	0.971	25.5	136	0.074
10.5	24	0.962	26.5	46	0.057
11.5	38	0.948	27.5	62	0.034
12.5	103	0.910	28.5	33	0.022
13.5	117	0.867	29.5	30	0.011
14.5	122	0.822	30.5	19	0.004
15.5	119	0.778	31.5	5	0.002
16.5	141	0.726	32.5	5	0.000



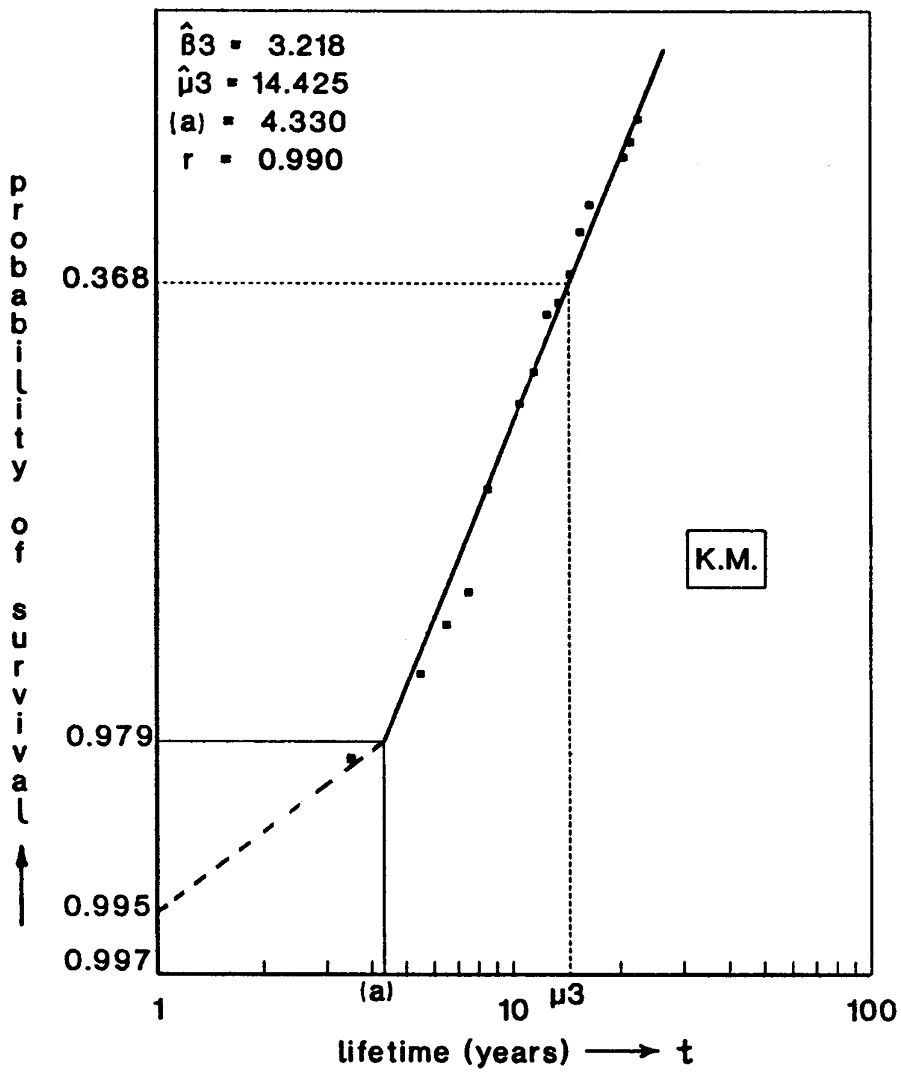
44-6 Cross Ties

T_j	d_j	$\hat{S}_j$
5.5	131	0.997
6.5	393	0.988
7.5	3058	0.918
8.5	2970	0.850
9.5	3582	0.768
10.5	5591	0.640
11.5	7033	0.479
12.5	12840	0.185
13.5	3582	0.103
14.5	2970	0.035
15.5	1529	0.000



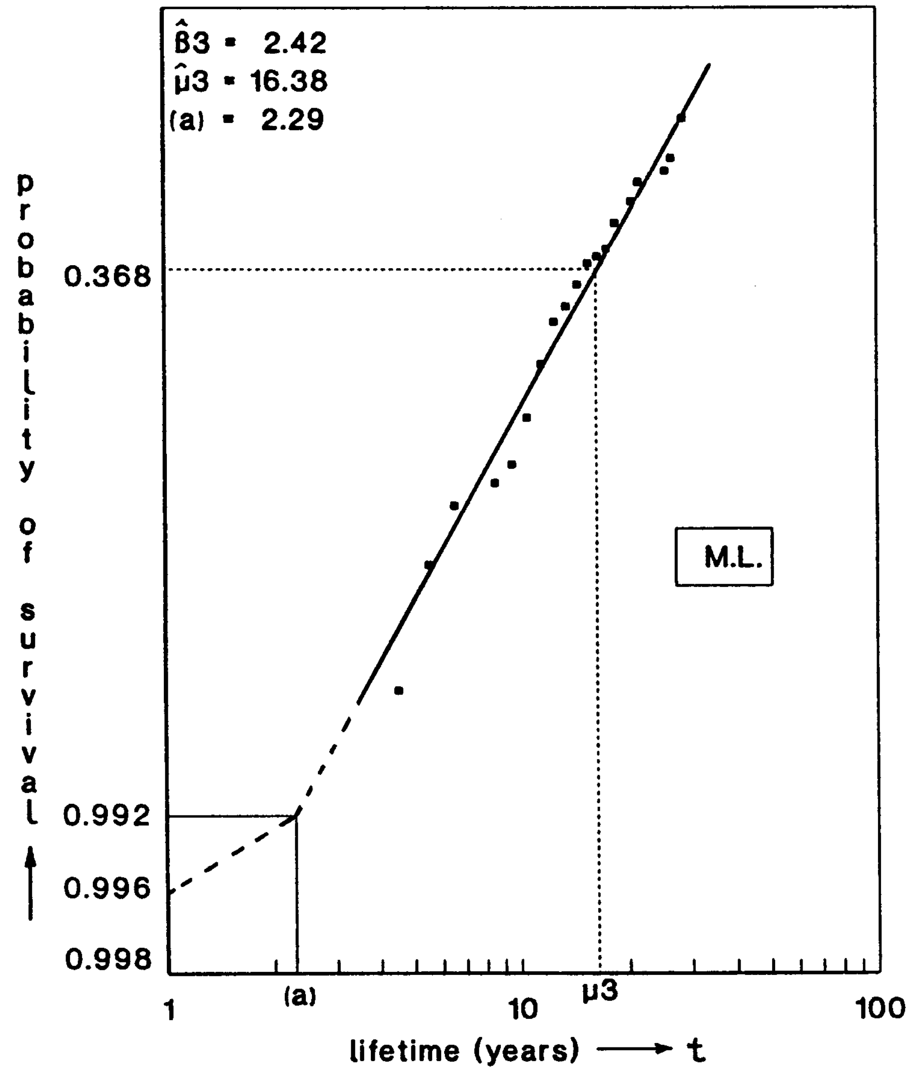
53-1 Rodger Ballast Cars

T_j	d_j	$\hat{S}_j$
1.5	5	0.993
2.5	5	0.987
3.5	6	0.979
4.5	5	0.972
5.5	6	0.965
6.5	4	0.959
7.5	5	0.953
8.5	5	0.946
9.5	10	0.933
10.5	19	0.908
11.5	7	0.899
12.5	19	0.874
13.5	27	0.839
14.5	27	0.803
15.5	11	0.789
16.5	22	0.760
17.5	27	0.725
18.5	18	0.701
19.5	18	0.678
20.5	42	0.623
21.5	168	0.402
22.5	67	0.314
23.5	46	0.253
24.5	81	0.147
25.5	42	0.092
26.5	13	0.075
27.5	11	0.060
28.5	18	0.037
31.5	28	0.000



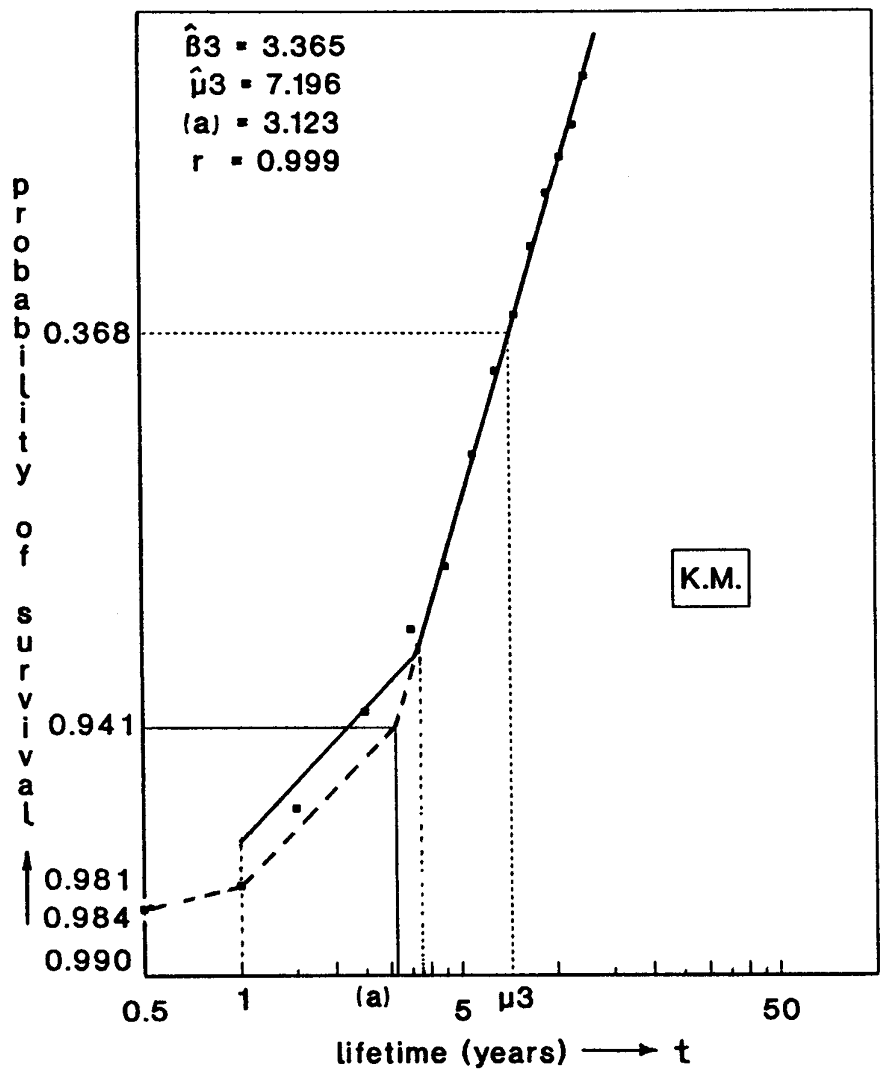
56-1 Corn Cultivators (1-row)

T_j	d_j	$\hat{S}_j$
3.5	1	0.982
5.5	1	0.964
6.5	1	0.946
7.5	1	0.929
8.5	5	0.839
10.5	8	0.696
11.5	4	0.625
12.5	9	0.464
13.5	2	0.429
14.5	5	0.339
15.5	7	0.214
16.5	4	0.143
20.5	5	0.054
21.5	1	0.036
22.5	1	0.018
30.5	1	0.000



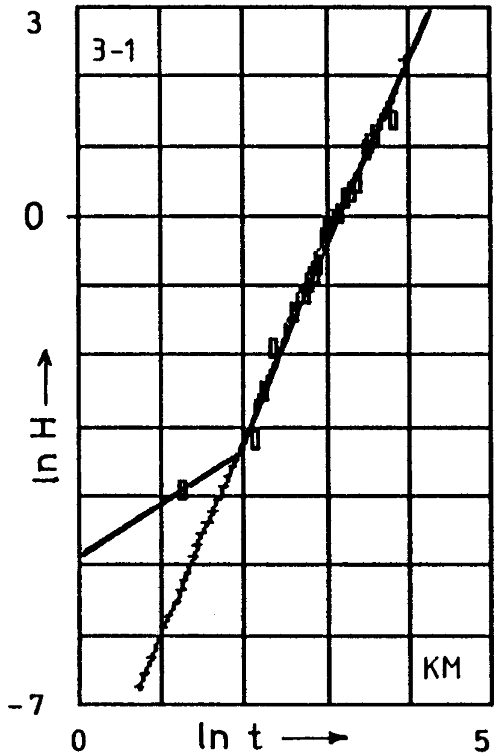
59-1 Grain binders (5- to 8-foot)

T_j	d_j	$\hat{S}_j$
4.5	1	0.977
5.5	2	0.930
6.5	2	0.884
8.5	1	0.860
9.5	1	0.837
10.5	7	0.674
11.5	1	0.651
12.5	5	0.535
13.5	2	0.488
14.5	3	0.419
15.5	3	0.349
16.5	1	0.326
17.5	1	0.302
18.5	3	0.223
20.5	3	0.163
21.5	2	0.116
25.5	1	0.093
26.5	1	0.070
28.5	2	0.023
30.5	1	0.000

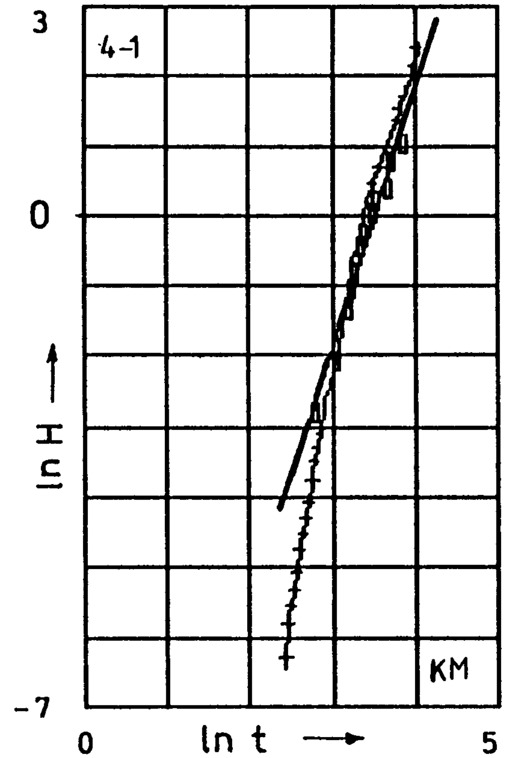


64-2 Automobiles

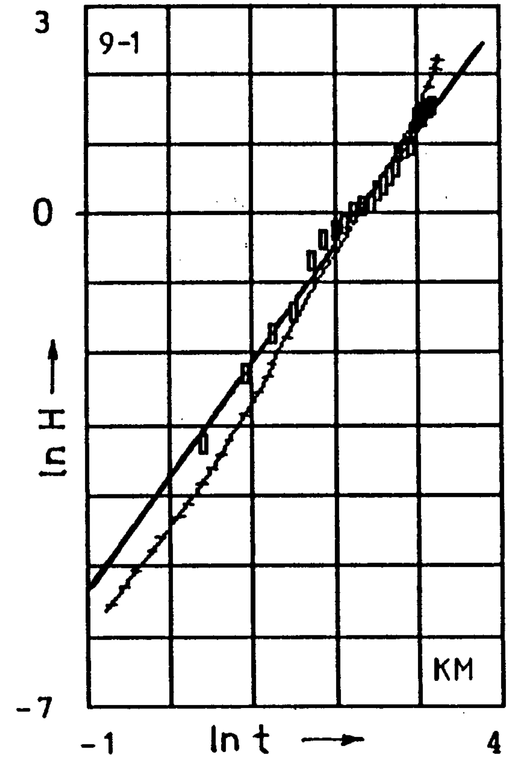
r_j	d_j	$\hat{S}_j$
0.5	51	0.984
1.5	50	0.968
2.5	99	0.936
3.5	151	0.888
4.5	182	0.829
5.5	533	0.659
6.5	600	0.467
7.5	457	0.320
8.5	515	0.156
9.5	282	0.065
10.5	112	0.029
11.5	55	0.012
12.5	31	0.002
13.5	6	0.000



Code: 3 - 1
WINFREY type curve: L^2
Parameters of MODEL:
$\hat{a}$ = 3.459 years
$\hat{\beta}_3$ = 2.646; $\hat{\mu}_3$ = 23.585 yrs

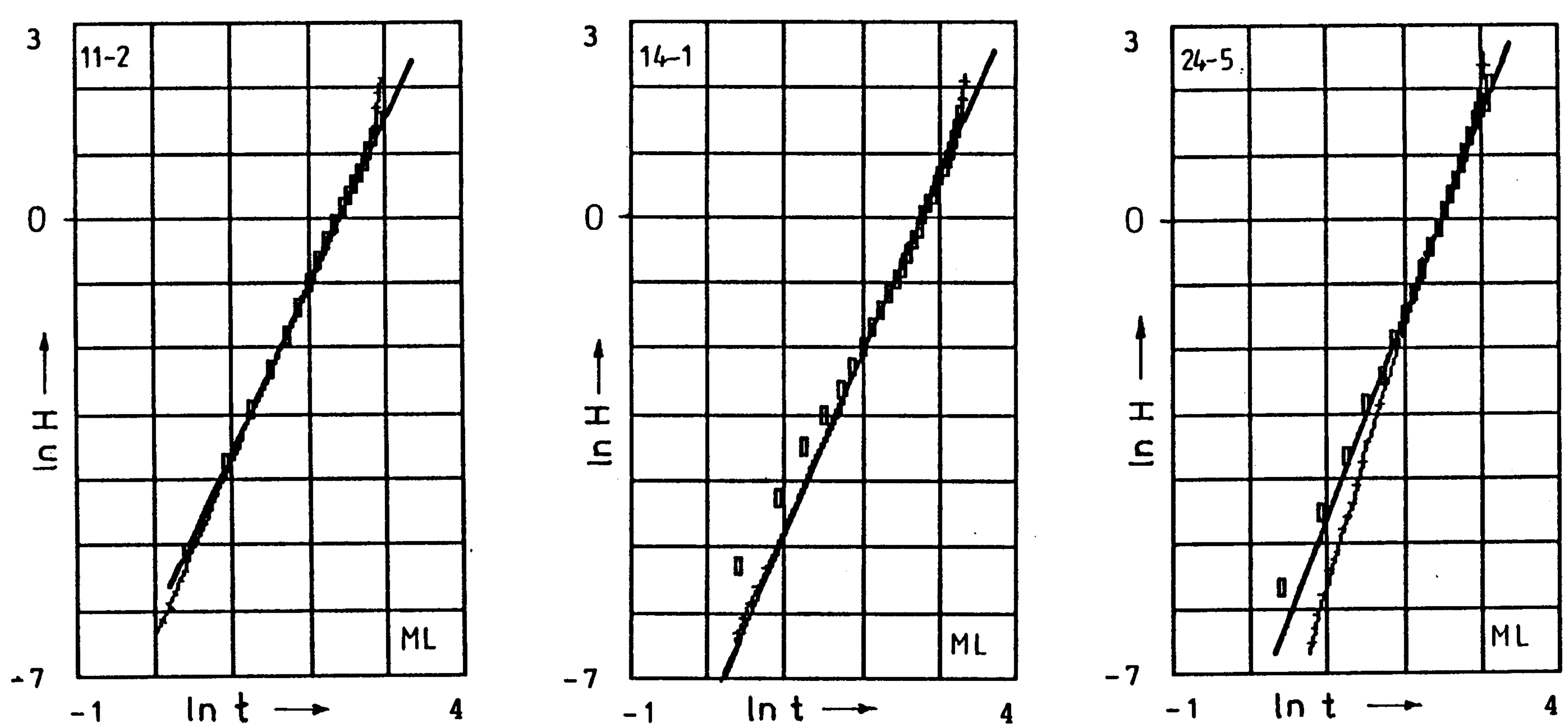


Code: 4 - 1
WINFREY type curve: L^4
Parameters of MODEL:
$\hat{a}$ = 9.798 years
$\hat{\beta}_3$ = 3.847; $\hat{\mu}_3$ = 33.717 yrs



Code: 9 -1
WINFREY type curve: L^1
Parameters of MODEL:
$\hat{a}$ = 0.308 years
$\hat{\beta}_3$ = 1.659; $\hat{\mu}_3$ = 9.767 yrs

Plots of empirical retirement data with related WINFREY type curves and (composite) WEIBULL survivor curves.

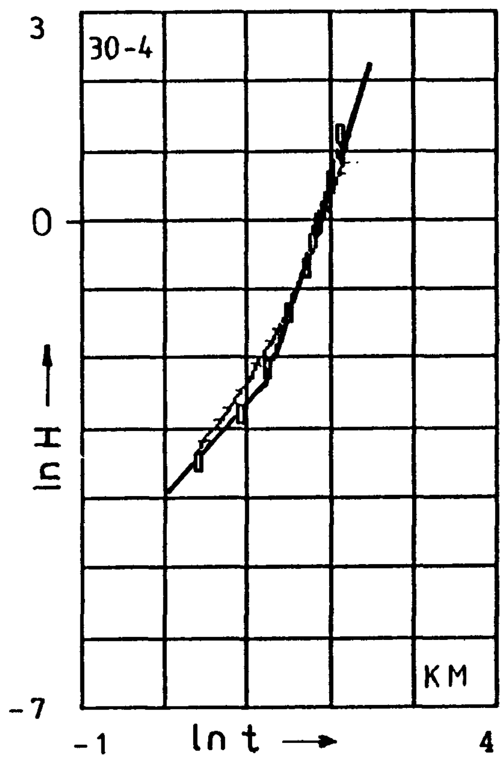


Code: 11 - 2
WINFREY type curve: S^1
Parameters of MODEL:
$\hat{a}$ = 2.52 years
$\hat{\beta}_3$ = 2.63; $\hat{\mu}_3$ = 10.90 yrs

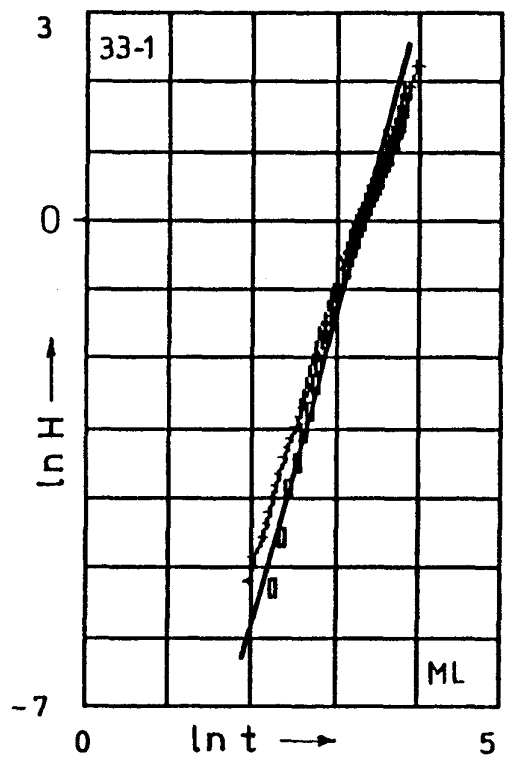
Code: 14 - 1
WINFREY type curve: S^1
Parameters of MODEL:
$\hat{a}$ = 3.27 years
$\hat{\beta}_3$ = 2.75; $\hat{\mu}_3$ = 15.93 yrs

Code: 24 - 5
WINFREY type curve: S^2
Parameters of MODEL:
$\hat{a}$ = 3.73 years
$\hat{\beta}_3$ = 3.13; $\hat{\mu}_3$ = 11.92 yrs

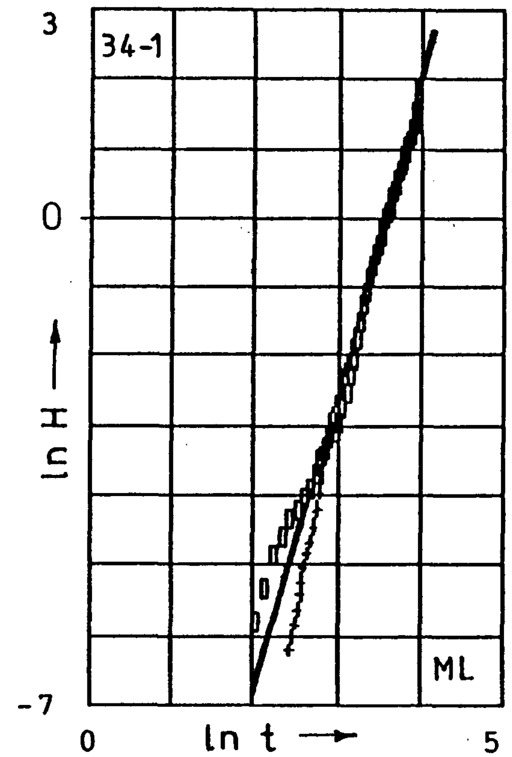
Plots of empirical retirement data with related WINFREY type curves and (composite) WEIBULL survivor curves.



Code: 30 - 4
WINFREY type curve: R^2
Parameters of MODEL:
$\hat{a}$ = 3.265 years
$\hat{\beta}_3$ = 3.763; $\hat{\mu}_3$ = 6.387 yrs

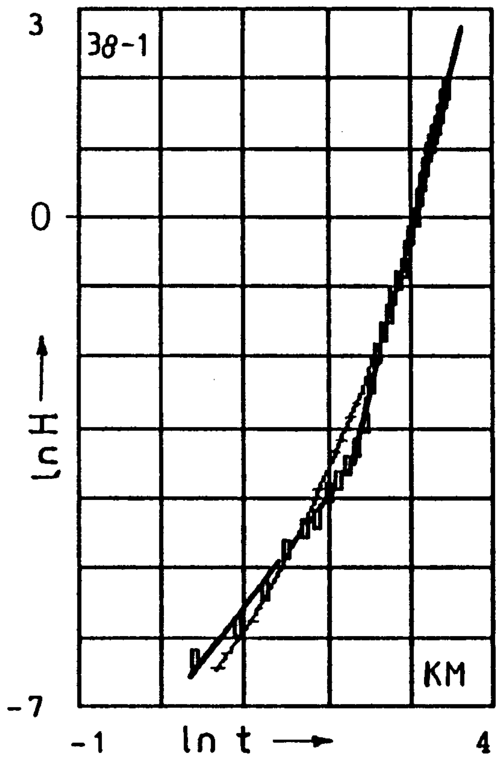


Code: 33 - 1
WINFREY type curve: L^3
Parameters of MODEL:
$\hat{a}$ = 7.98 years
$\hat{\beta}_3$ = 3.66; $\hat{\mu}_3$ = 27.93 yrs

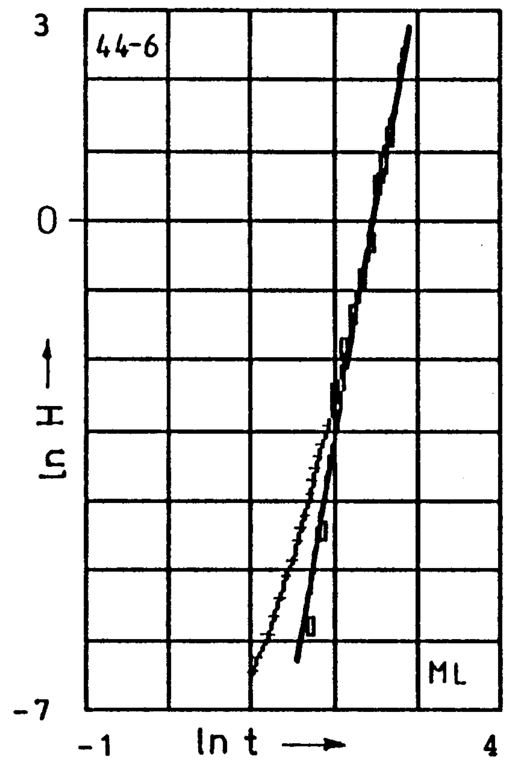


Code: 34 - 1
WINFREY type curve: S^3
Parameters of MODEL:
$\hat{a}$ = 12.25 years
$\hat{\beta}_3$ = 4.31; $\hat{\mu}_3$ = 36.17 yrs

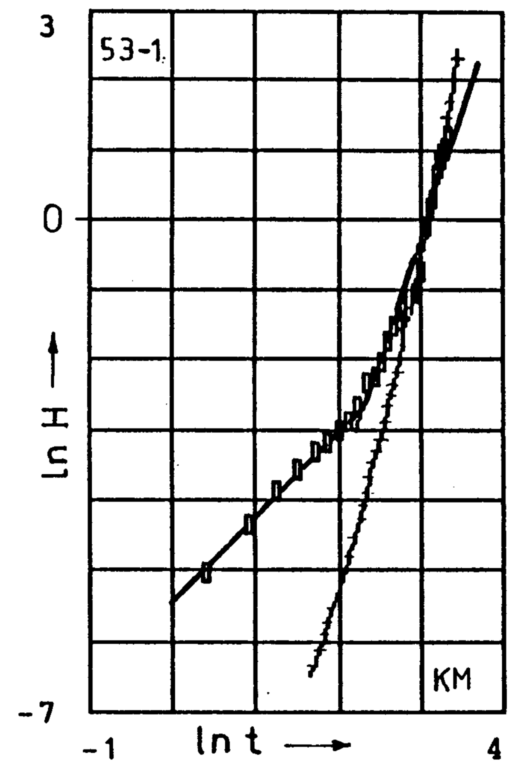
Plots of empirical retirement data with related WINFREY type curves and (composite) WEIBULL survivor curves.



Code: 38 - 1
WINFREY type curve: R^3
Parameters of MODEL:
$\hat{a}$ = 9.014 years
$\hat{\beta}_3$ = 4.581; $\hat{\mu}_3$ = 21.133 yrs

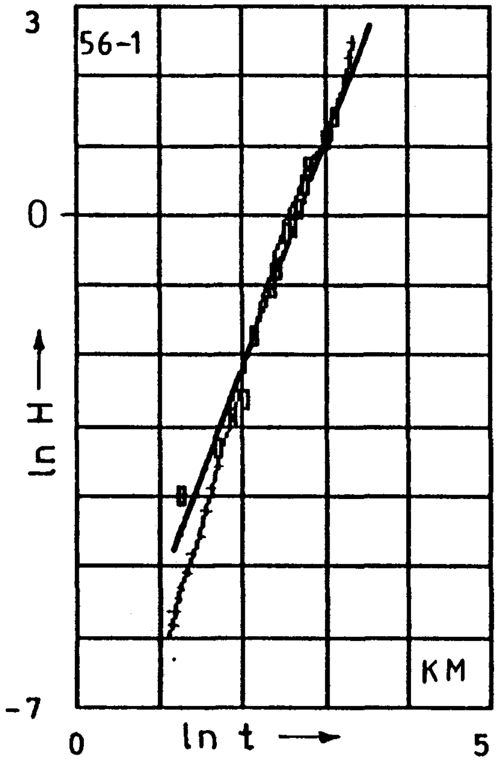


Code: 44 - 6
WINFREY type curve: R^4
Parameters of MODEL:
$\hat{a}$ = 7.35 years
$\hat{\beta}_3$ = 6.21; $\hat{\mu}_3$ = 11.80 yrs

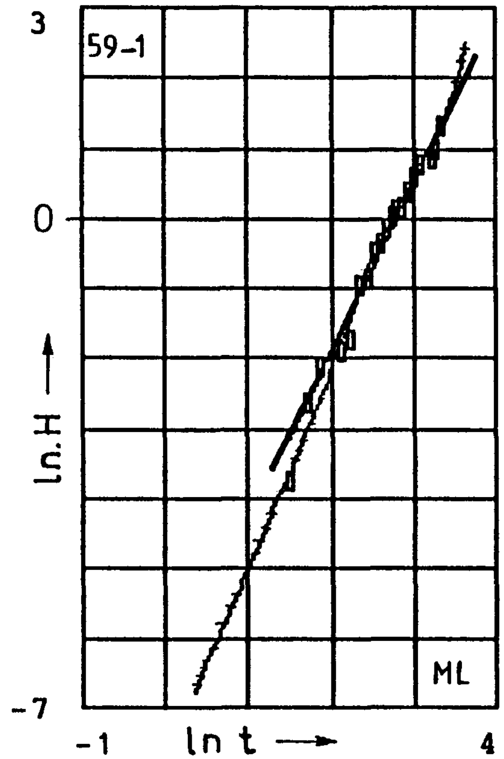


Code: 53 - 1
WINFREY type curve: R^4
Parameters of MODEL:
$\hat{a}$ = 6.311 years
$\hat{\beta}_3$ = 3.477; $\hat{\mu}_3$ = 21.976 yrs

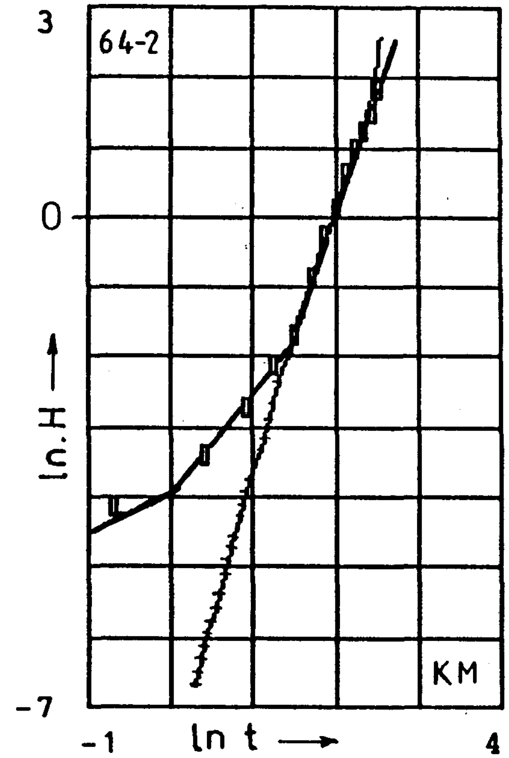
Plots of empirical retirement data with related WINFREY type curves and (composite) WEIBULL survivor curves.



Code: 56 - 1
WINFREY type curve: L^3
Parameters of MODEL:
$\hat{a}$ = 4.330 years
$\hat{\beta}_3$ = 3.218; $\hat{\mu}_3$ = 14.425 yrs

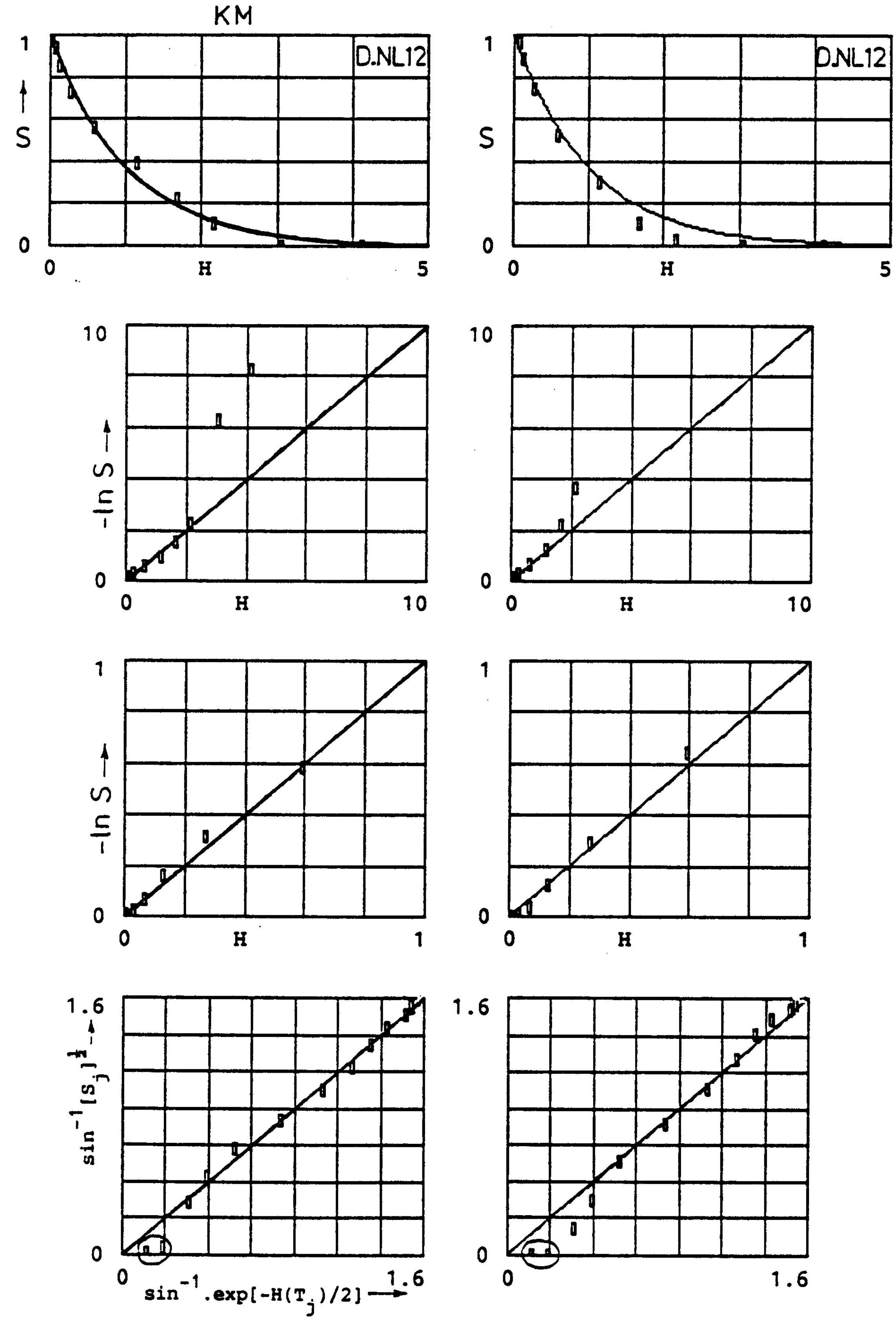


Code: 59 - 1
WINFREY type curve: L^2
Parameters of MODEL:
$\hat{a}$ = 2.29 years
$\hat{\beta}_3$ = 2.42; $\hat{\mu}_3$ = 16.38 yrs

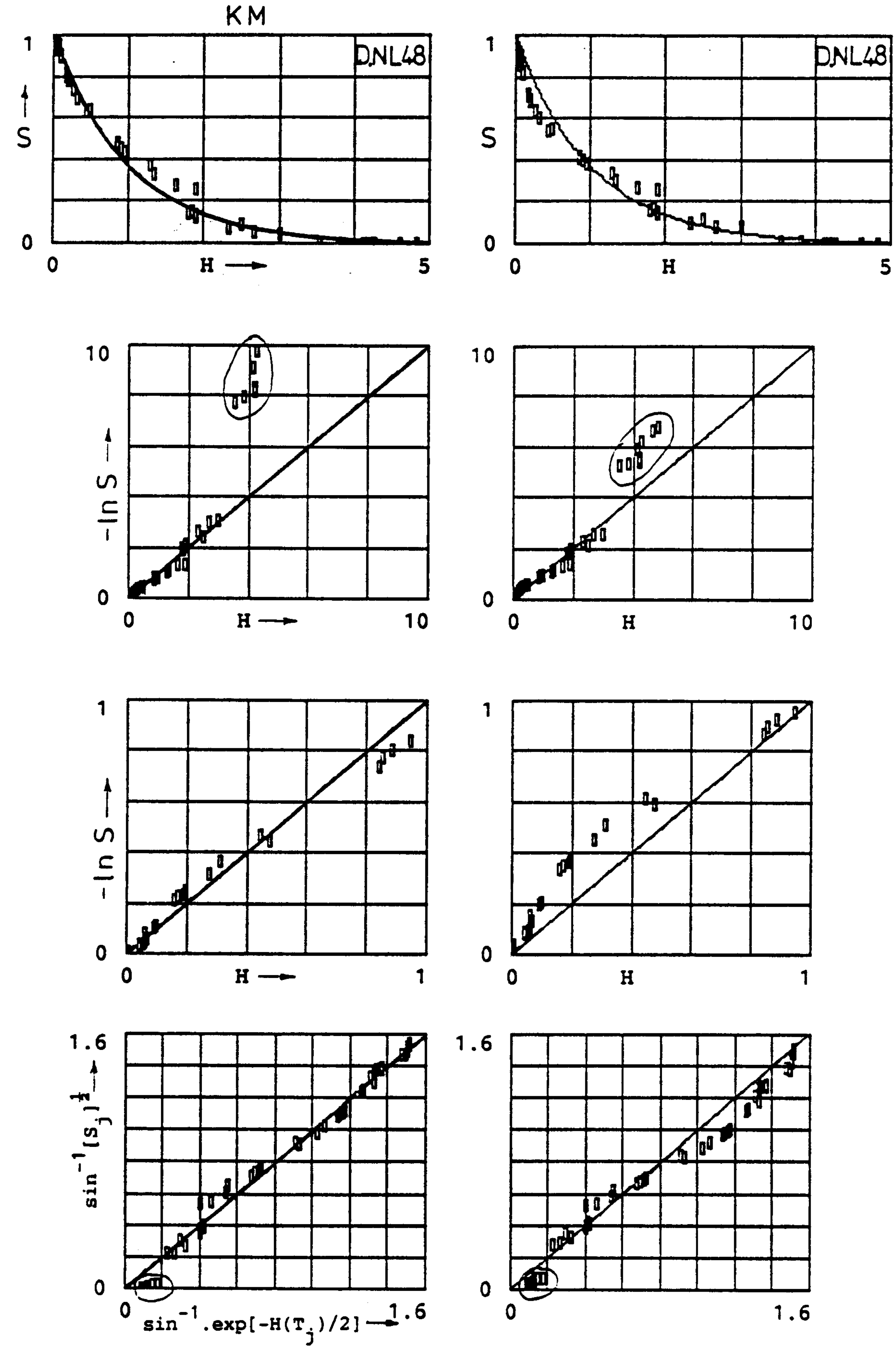


Code: 64 - 2
WINFREY type curve: S^2
Parameters of MODEL:
$\hat{a}$ = 3.123 years
$\hat{\beta}_3$ = 3.365; $\hat{\mu}_3$ = 7.196 yrs

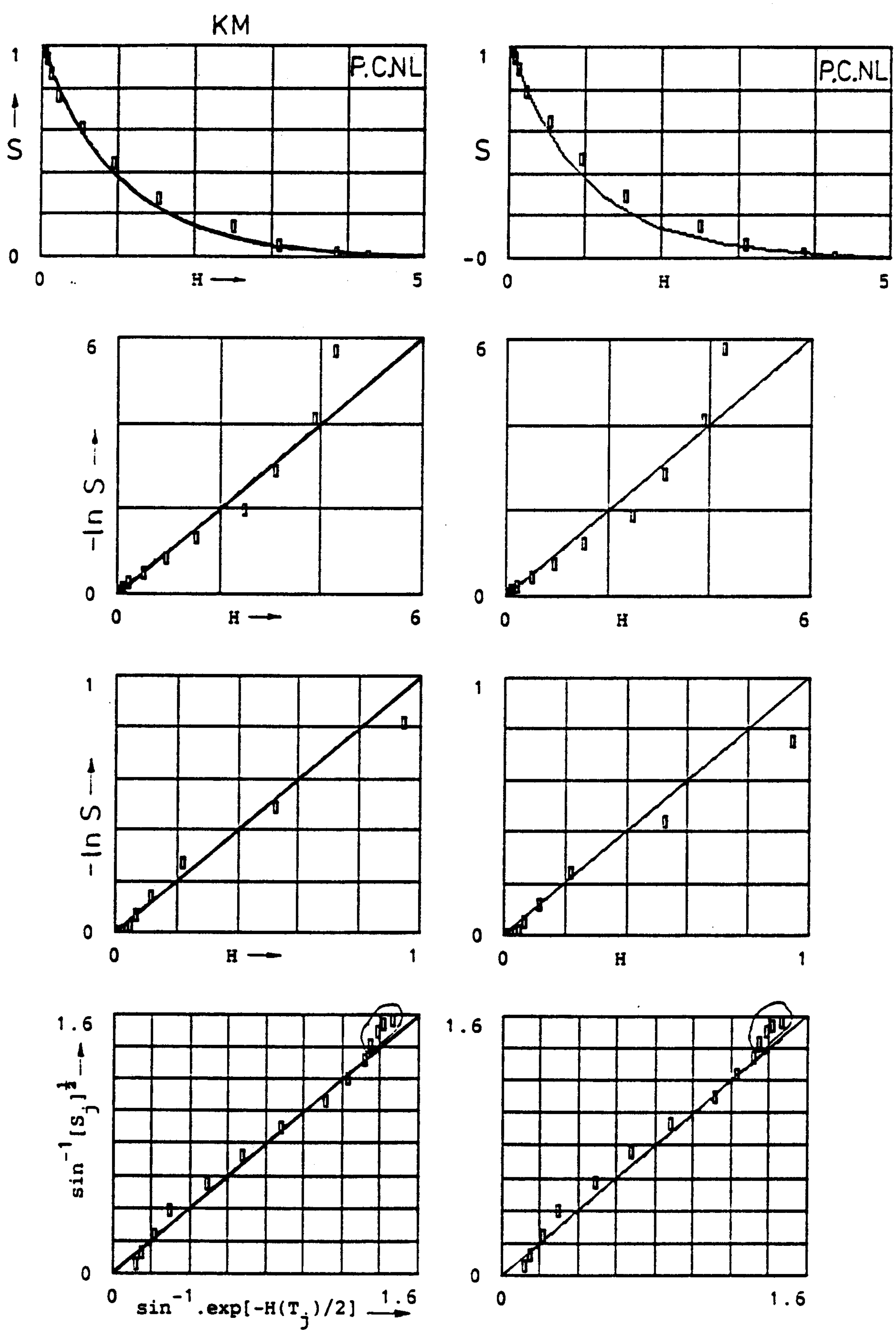
Plots of empirical retirement data with related WINFREY type curves and (composite) WEIBULL survivor curves.



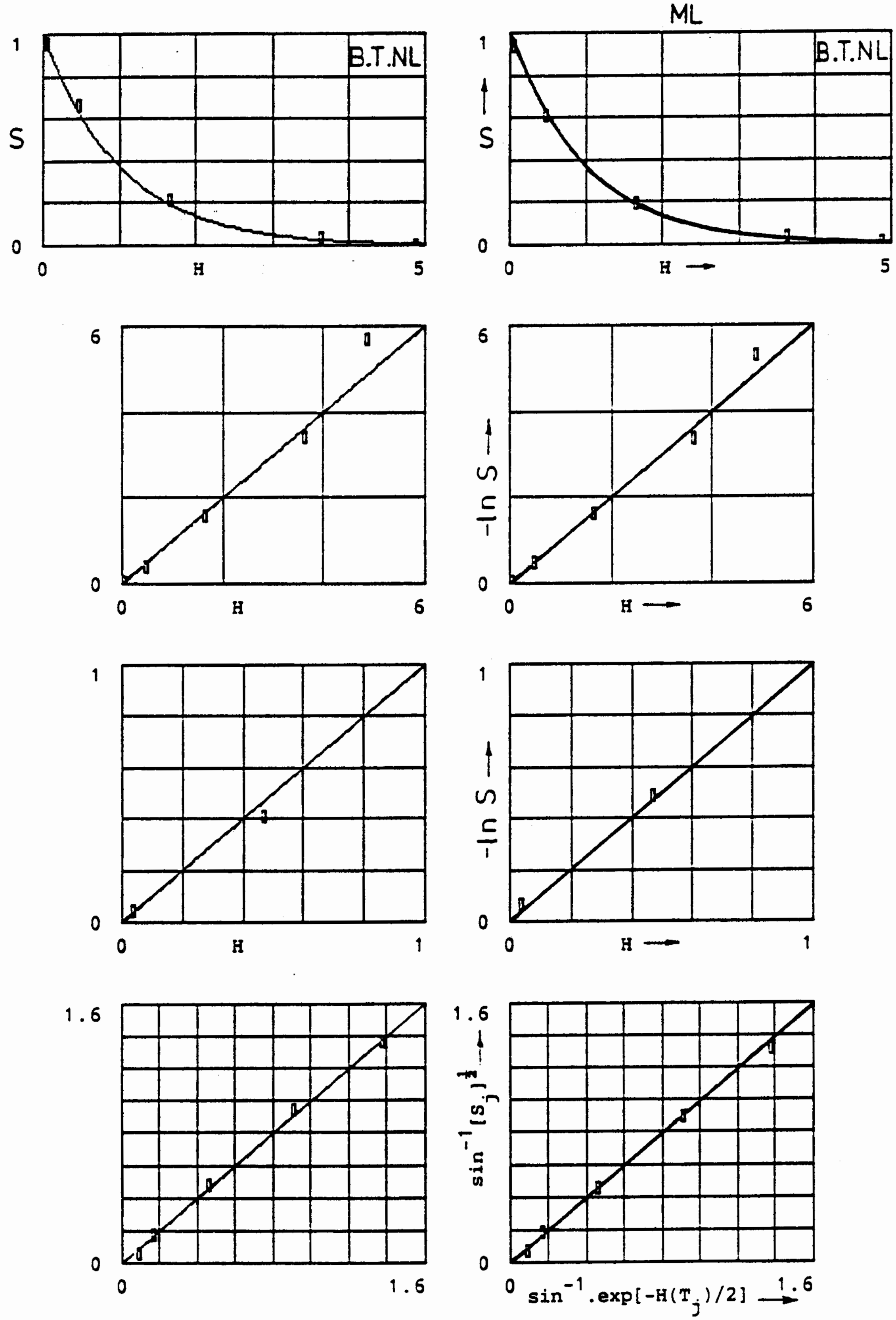
Plot of H-residuals: Dwellings NL (12 points), Code D.NL12



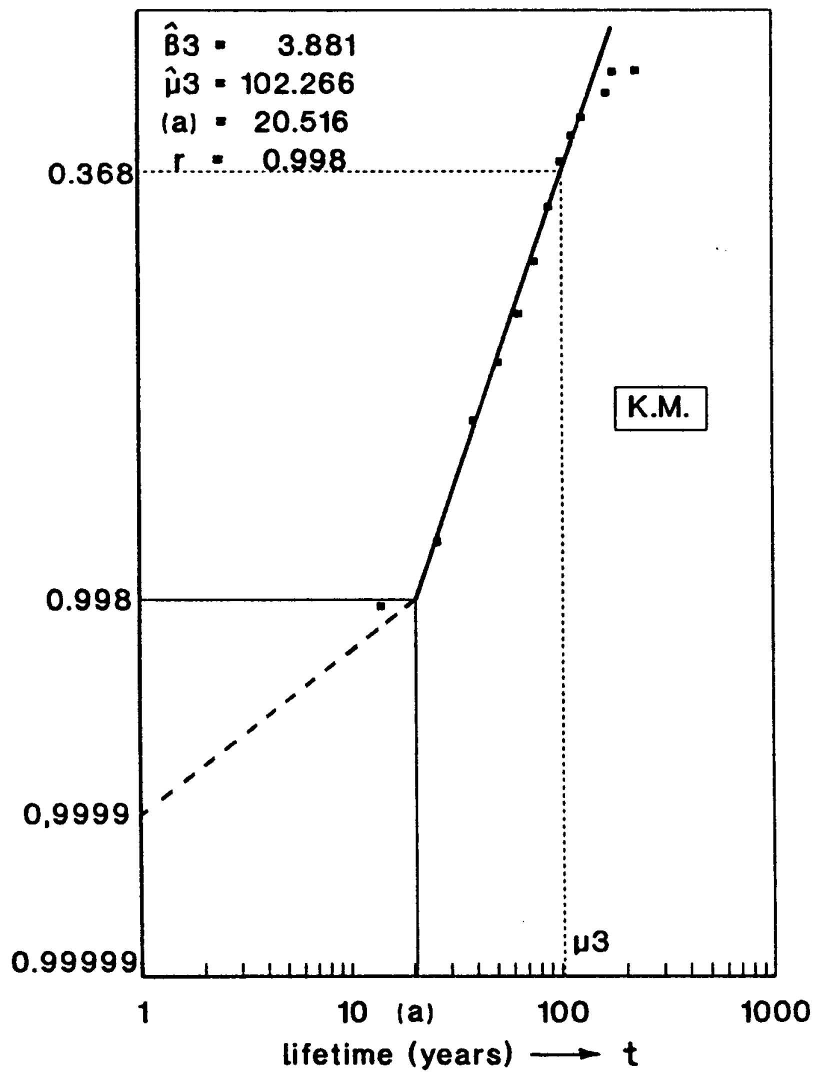
Plot of H-residuals: Dwellings NL (48 points), Code D.NL48



Plot of H-residuals: Passenger cars NL, Code P.C.NL.

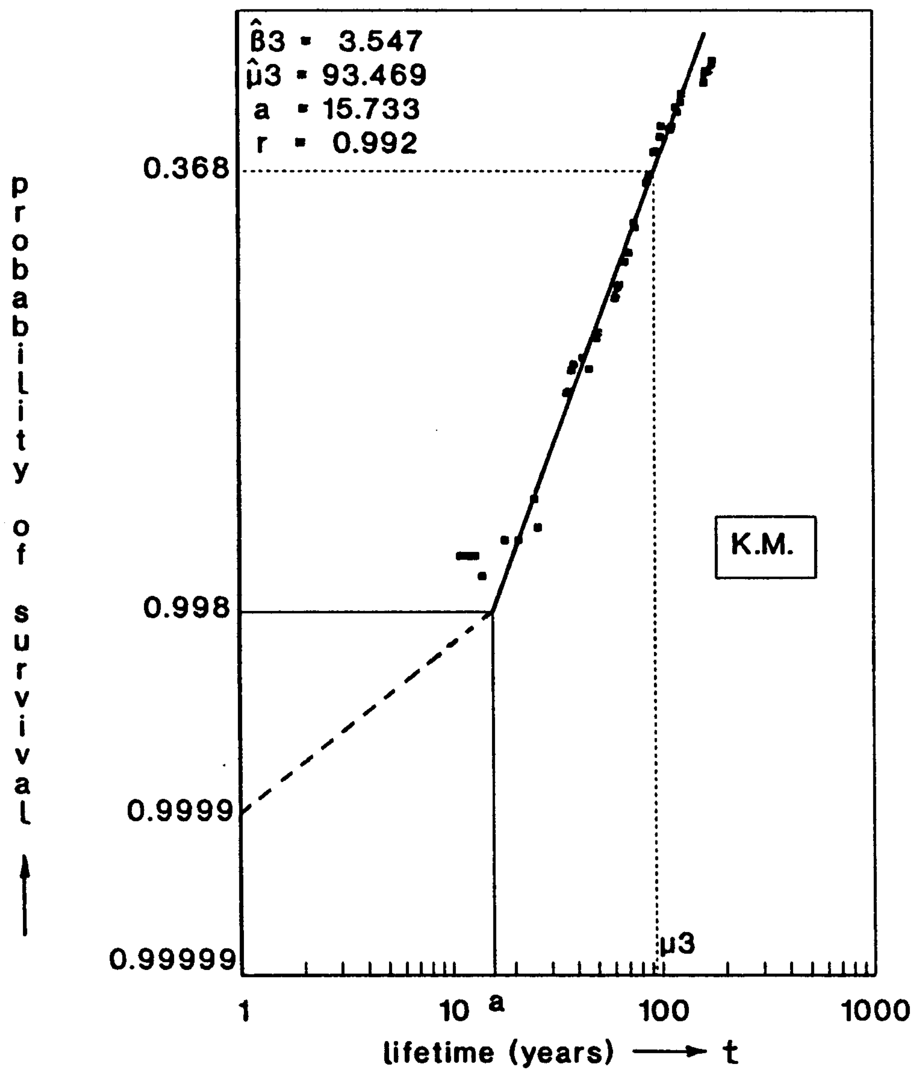


Plot of H-residuals: Bus tyres (The Hague, NL), Code B.T.NL.



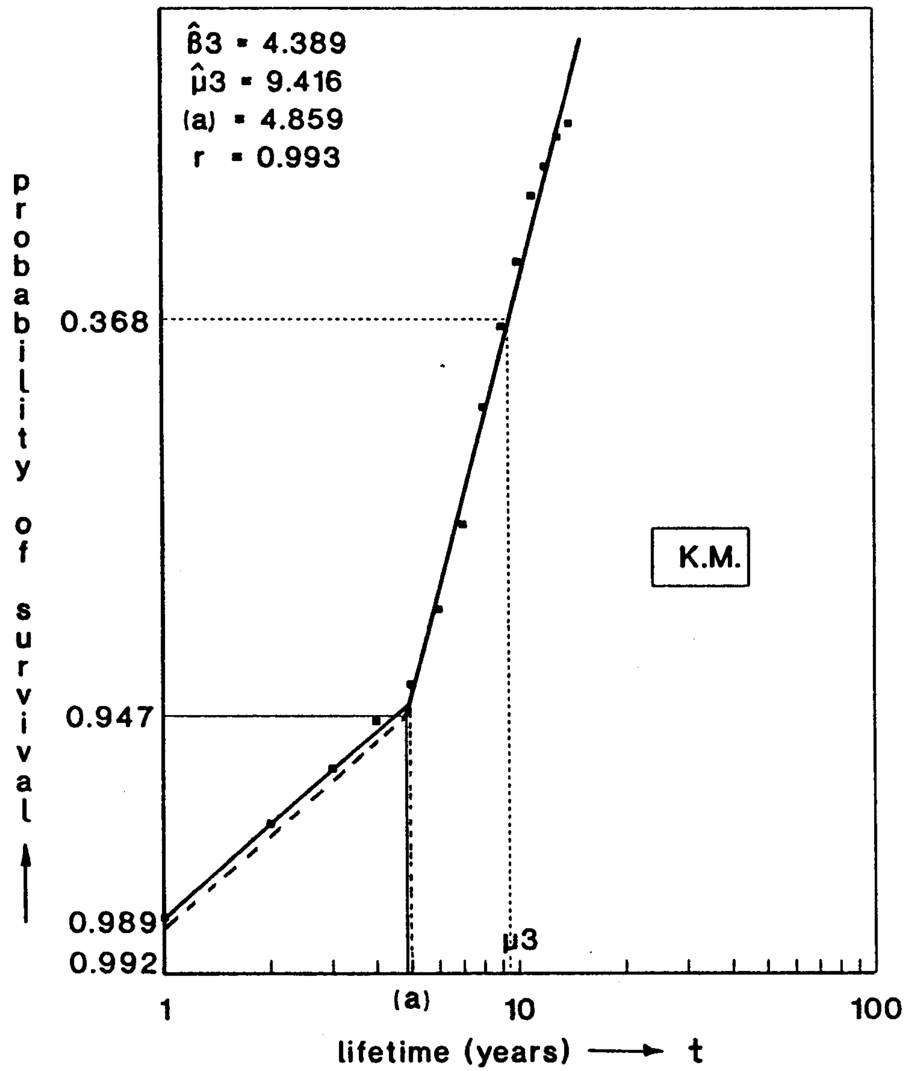
Dwellings (composition)

τ_j	d_j	$\hat{S}_j$
14	15	0.998
26	25	0.995
39	181	0.972
51	277	0.938
64	460	0.880
76	940	0.763
89	1687	0.552
101	1875	0.318
114	1031	0.189
126	585	0.116
164	545	0.047
176	250	0.016
190	129	0.000



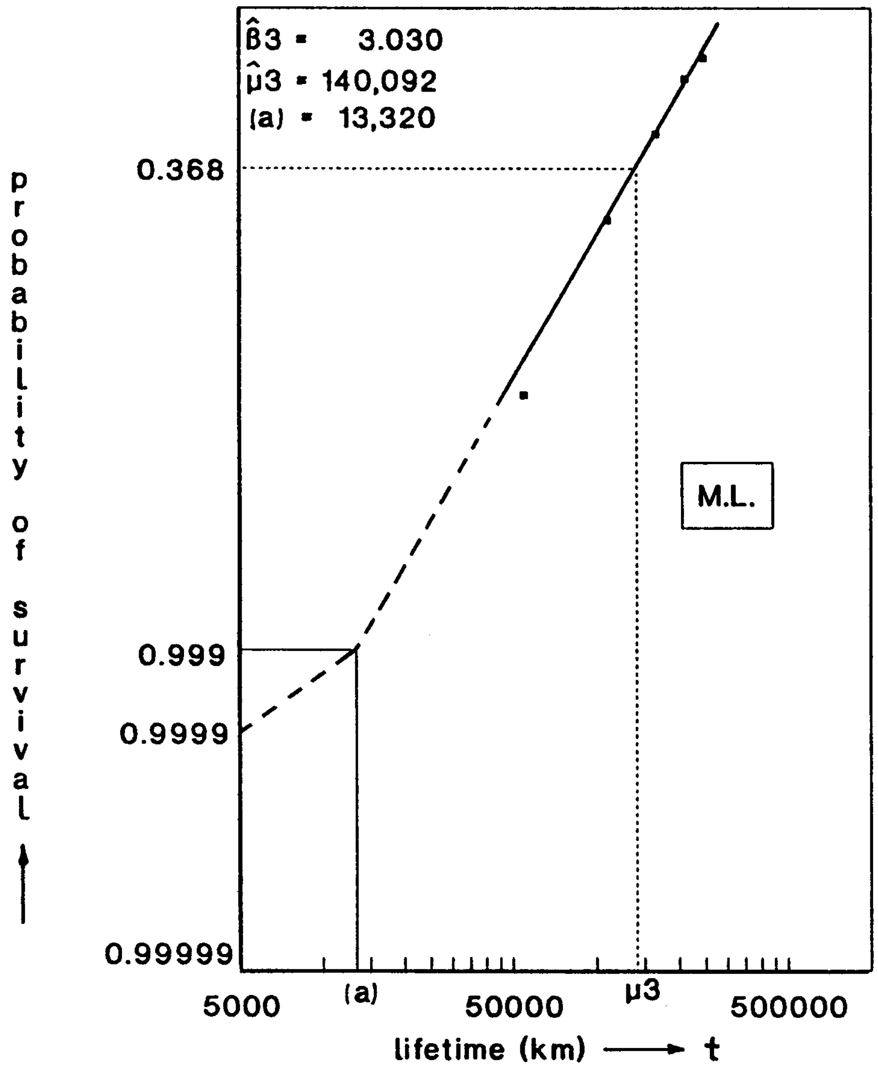
Dwellings 1961-1976

T_j	d_j	$\hat{S}_j$	T_j	d_j	$\hat{S}_j$
11	4	0.996	86	419	0.432
12	4	0.996	87	419	0.425
13	4	0.996	88	416	0.412
14	3	0.997	89	433	0.388
18	5	0.995	93	489	0.273
21	5	0.995	96	470	0.264
25	9	0.991	100	426	0.196
26	6	0.994	101	490	0.150
36	36	0.960	111	272	0.160
37	37	0.959	112	259	0.166
38	51	0.945	113	263	0.149
39	57	0.940	114	237	0.151
43	61	0.934	118	191	0.082
46	52	0.943	121	166	0.098
50	77	0.914	125	127	0.069
51	87	0.907	126	101	0.049
61	109	0.851	161	131	0.029
62	115	0.844	162	144	0.022
63	117	0.828	163	134	0.015
64	119	0.821	164	136	0.015
68	172	0.762	168	66	0.016
71	209	0.734	171	84	0.014
75	292	0.622	175	68	0.010
76	267	0.640	176	41	0.008



Passenger cars CBS

τ_j	d_j	$\hat{S}_j$
1	12	0.988
2	12	0.976
3	11	0.965
4	15	0.950
5	15	0.935
6	45	0.890
7	87	0.803
8	210	0.593
9	205	0.388
10	170	0.218
11	135	0.083
12	38	0.045
13	24	0.021
14	7	0.014
>14	14	0.000



Bus tyres H.T.M.

T_j	d_j	$\hat{S}_j$
55,000	10	0.962
110,000	89	0.625
165,000	114	0.193
210,000	44	0.027
245,000	5	0.008
270,000	2	0.000

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